Spring 2005 Math 152
10 Infinite Sequences and Series
10.6 Representations of Functions as Power Series
Wed, 06/Apr ©2005, Art Belmonte

Summary

THEOREM

A power series \( f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \) having a radius of convergence \( R > 0 \) is differentiable and integrable on the interior of its interval of convergence; i.e., \( |x - a| < R \) or \( (a - R, a + R) \). Essentially, we may differentiate or integrate term-by-term.

\[
f'(x) = \frac{d}{dx} \left( \sum_{n=0}^{\infty} c_n (x - a)^n \right) = \sum_{n=1}^{\infty} n c_n (x - a)^{n-1}
\]

\[
\int f(x) \, dx = \int \sum_{n=0}^{\infty} c_n (x - a)^n \, dx = C + \sum_{n=0}^{\infty} \frac{c_n (x - a)^{n+1}}{n+1}
\]

The radii of convergence of the derivative and antiderivative series representations are each equal to \( R \). (NOTE: The intervals of convergence of these representations may not be the same be the same as the interval of convergence \( I \) of the original power series. At the endpoints of \( I \), if any, convergence must still be checked.)

A way to obtain the series representation of a function

Starting with the geometric series \( \sum_{n=0}^{\infty} x^n \), which converges to \( \frac{1}{1-x} \) for \( |x| < 1 \) and diverges otherwise, we may be able to obtain series representations for other functions by algebraic manipulation, differentiation or indefinite integration, and/or use of the preceding theorem.

While this is by no means systematic, it will have to do until a more sure-fire technique is introduced in the next section: Taylor and Maclaurin series.

Hand Examples

Example A

Find a power series representation for \( f(x) = \frac{2}{3x + 4} \) and determine its radius and interval of convergence.

Solution

- Use algebraic manipulation together with the Geometric Series Theorem (GST).

\[
\frac{2}{3x + 4} = \frac{2}{4 \left(1 - \left(-\frac{3}{4}x\right)\right)} = \frac{1}{2} \left(1 - \left(-\frac{3}{4}x\right)\right) = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{3}{4}x\right)^n, \quad \text{for } \left|-\frac{3}{4}x\right| < 1 \text{ via GST}
\]

We require \( \left|-\frac{3}{4}x\right| < 1 \) or \( |x| < \frac{4}{3} \). Therefore, \( R = \frac{4}{3} \) and the center of \( I \) is \( x = 0 \).

- At \( x = -\frac{4}{3} \), the left endpoint of \( I \), the series is \( \sum \frac{1}{2} \), which diverges by the Test for Divergence since \( \lim_{n \to \infty} \frac{1}{2^n} = 0 \).

- At \( x = \frac{4}{3} \), the right endpoint of \( I \), the series is \( \sum \left(-\frac{1}{2}\right) \), which diverges by oscillation.

- Therefore, \( I = \left(-\frac{4}{3}, \frac{4}{3}\right) \).

Example B

Find a power series representation for \( f(x) = \frac{x}{x^2 - 3x + 2} \) and determine its radius and interval of convergence.

Solution

- First split the expression into a sum of partial fractions.

\[
\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}
\]

\[
x = A(x-2) + B(x-1)
\]

Thus \( A + B = 1 \) and \(-2A - B = 0\). So \( B = -2A \) whence \(-A = 1 \) or \( A = -1 \) and \( B = 2 \). Therefore,

\[
\frac{x}{x^2 - 3x + 2} = \frac{-1}{x-1} + \frac{2}{x-2}
\]

- Now use algebraic manipulation and the GST.

\[
\frac{1}{1-x} - \frac{2}{2-x} = \frac{1}{1-x} - \frac{1}{1-\frac{1}{2}x}
\]

If \( |x| < 1 \) and \( \left|\frac{1}{2}\right| < 1 \) via GST:

\[
\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n x^n = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^n}\right) x^n
\]

\[
\sum_{n=1}^{\infty} \left(1 - \frac{1}{2^n}\right) x^n
\]
We require both $|x| < 1$ and $\frac{1}{|x|} < 1$; that is, both $|x| < 1$ and $|x| < 2$. In other words, we need $|x| < 1$. Therefore, $R = 1$ and the center of $I$ is $x = 0$.

- At $x = -1$, the left endpoint of $I$, the series is $\sum a_n = \sum (1 - \frac{1}{2^k}) (-1)^n$, which diverges by the Test for Divergence since $\lim a_n \neq 0$.

- At $x = 1$, the right endpoint of $I$, the series is $\sum a_n = \sum (1 - \frac{1}{2^k})$, which diverges by the Test for Divergence since $\lim a_n \neq 0$.

- Therefore, $I = (-1, 1)$.

622/6

Find a power series representation for $f(x) = \frac{1}{x^4 + 16}$ and determine its radius and interval of convergence.

Solution

- Use algebraic manipulation and the GST.

\[
\frac{1}{x^4 + 16} = \frac{1}{16 \left( 1 - \left( -\frac{x^4}{16} \right) \right)} = \frac{1}{16} \sum_{n=0}^{\infty} \left( -\frac{x^4}{16} \right)^n, \text{ if } \left| -\frac{x^4}{16} \right| < 1
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{16^{n+1}}
\]

We require $\left| -\frac{x^4}{16} \right| < 1$ or $|x|^4 < 16$ or $|x| < 2$. Therefore, $R = 2$ and the center of $I$ is $x = 0$.

- At $x = -2$, the left endpoint of $I$, the series is $\sum \frac{(-1)^n}{16}$, which diverges by oscillation.

- At $x = 2$, the right endpoint of $I$, the series is also $\sum \frac{(-1)^n}{16}$, which diverges by oscillation.

- Therefore, $I = (-2, 2)$.

622/10

Find a power series representation for $f(x) = \ln(1 + x)$ and determine its radius and interval of convergence.

Solution

- Note that $\ln(1 + x)$ is an antiderivative of $\frac{1}{1 + x}$.

\[
\ln(1 + x) = \int \frac{1}{1 + x} \, dx = \int \frac{1}{1 - (-x)} \, dx = \int \sum_{n=0}^{\infty} (-x)^n \, dx, \quad \text{if } |x| < 1
\]

\[
= C + \sum_{n=0}^{\infty} \frac{(-1)^n \int x^n \, dx}{n + 1}
\]

\[
= C + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}
\]

\[
0 = \ln(1 + 0) = C
\]

Thus $\ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$.

We require $|x| < 1$ or $|x| < 1$. Therefore, $R = 1$ and the center of $I$ is $x = 0$.

- At $x = -1$, the left endpoint of $I$, the series is $- \sum \frac{1}{k}$, which diverges to $-\infty$ since the $p$-series $\sum \frac{1}{k}$ diverges to $\infty$ ($p = 1 \leq 1$).

- At $x = 1$, the right endpoint of $I$, the series is $\sum \frac{(-1)^{k-1}}{k}$, which converges by the AST since $\frac{1}{k} \downarrow 0$.

- Therefore, $I = (-1, 1)$.

622/11

Find a power series representation for $f(x) = \frac{1}{(1 + x)^3}$ and determine its radius and interval of convergence.

Solution

- By 622/10 we have $\ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$ for $x \in (-1, 1]$. Hence

\[
(1 + x)^{-1} = \frac{1}{1 + x}
\]

\[
= \frac{d}{dx} (\ln(1 + x))
\]

\[
= \frac{d}{dx} \left( \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k} \right)
\]

\[
= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{k-1}}{k}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n x^n},
\]

which holds for $x \in (-1, 1)$ but clearly not at the endpoints of $I$. Thus $(1 + x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$ for $|x| < 1$. 

determine its radius and interval of convergence.

For the series to converge by the Ratio Test, we need

\[
\left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{(-1)^{k+1} x^k}{(-1)^k x^{k-1}} \right| = \left| \frac{x}{1} \right| = |x| < 1.
\]

Hey campers, the third time’s the charm!

\[
(1 + x)^{-3} = \frac{d}{dx} \left( \frac{1}{1 + x} \right)^2 = \frac{d}{dx} \left( -\sum_{n=0}^{\infty} (-1)^n x^n \right) = \sum_{k=1}^{\infty} (-1)^k (k+1) x^k - 2.
\]

We conclude that

\[
\frac{1}{(1 + x)^3} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2) (n+1) x^n}{2}, \quad \text{for } |x| < 1.
\]

622/12

Find a power series representation for \( f(x) = x \ln(1 + x) \) and determine its radius and interval of convergence.

### Solution

- By 622/10 we have

\[
x \ln(1 + x) = x \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{k+1}}{k}.
\]

- For the series to converge by the Ratio Test, we need

\[
\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{(-1)^{k+1} x^k}{(-1)^k x^{k-1}} \right| = \lim_{k \to \infty} \left| \frac{x}{1} \right| = |x| < 1.
\]

Therefore, \( R = 1 \) and the center of \( I \) is \( x = 0 \).

- At \( x = -1 \), the left endpoint of \( I \), the series is \( \sum_{k} \frac{1}{k} \), the divergent harmonic series.

- At \( x = 1 \), the right endpoint of \( I \), the series is \( \sum_{k} (-1)^{k-1} \frac{1}{k} \), which converges by the AST since \( \frac{1}{k} \downarrow 0 \).

Therefore, \( I = (-1, 1] \).

622/22

Evaluate the indefinite integral \( \int \tan^{-1} (x^2) \, dx \) as a power series.

#### Solution

- From Example 7 on page 621 of Stewart we have

\[
\tan^{-1} \theta = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{2n+1}, \quad \text{for } |\theta| \leq 1.
\]

- Thus \( \tan^{-1} (x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}, \quad \text{for } |x| \leq 1. \)

- Hence

\[
\int \tan^{-1} (x^2) \, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}, \quad \text{for } |x| < 1.
\]

622/28

Show that the function

\[
f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}
\]

is a solution of the differential equation

\[
f''(x) + f(x) = 0.
\]

#### Solution

- We have

\[
f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}.
\]

- In turn,

\[
f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1) x^{2n-2}}{(2n-1)!}
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{2(2n-2)!}
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1) x^{2(n-1)}}{(2(n-1))!}
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k}}{(2k)!}
\]

\[
= - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.
\]

- Hence

\[
f''(x) + f(x) = \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \right) + \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \right) = 0.
\]

Therefore, \( f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \) is a solution of the stated differential equation.
MATLAB Examples

s622x16

Find a power series representation for \( f(x) = \frac{1}{x^2 + 25} \) and graph \( f(x) \) and several partial sums \( s_n(x) \) on the same plot.

Solution

Use algebraic manipulation and the GST.

\[
\frac{1}{x^2 + 25} = \frac{1}{25} \left( 1 - \frac{1}{25} x^2 \right) = \frac{1}{25} \left( 1 - \frac{1}{(2n+1)(4n+3)} \right)
\]

\[
= \frac{1}{25} \sum_{n=0}^{\infty} \left( -\frac{1}{25} \right)^n x^{2n}, \text{ if } \left| -\frac{1}{25} x^2 \right| < 1
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{25^{n+1}}
\]

We require \( \left| -\frac{1}{25} x^2 \right| < 1 \) or \( |x|^2 < 25 \) or \(|x| < 5\). Therefore, \( R = 5 \) and the center of \( I \) is \( x = 0 \). Each successive partial sum of the series representation approximates the graph of \( f(x) \) more closely.

622/24

Use a power series to approximate the definite integral

\[
\int_0^{1/2} \tan^{-1}(x^2) \, dx
\]

to six decimal places (\( \epsilon = 5 \times 10^{-7} \)).

Solution

- Using the alternating series from 622/22, we have

\[
\int_0^{1/2} \tan^{-1}(x^2) \, dx = \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)(4n+3)} \right) \bigg|_{x=0}^{x=1/2}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(4n+1)}
\]

- By the ASET, we need

\[
|R_N| \leq |a_{N+1}| = \frac{1}{(2N+3)(4N+7)2^{4N+7}} \leq 5 \times 10^{-7}
\]

which implies \( N \geq 1.84 \). So choose \( N = 2 \), for which the partial sum of the series is

\[
\sum_{n=0}^{2} \frac{(-1)^n}{2^n(4n+1)} \approx 0.0413.
\]

% Stewart 622/24

% syms N

N = 0:3;

v = 1 ./ ( (2*N+3) * (4*N+7) * 2^(4*N+7) ); pretty(v)

\[
\begin{array}{c|c}
N & v \\
\hline
0 & 3.72e^{-4} \\
1 & 8.88e^{-6} \\
2 & 2.91e^{-7} \\
3 & 1.12e^{-8} \\
\end{array}
\]

% syms n

a = (-1)^n / ( (2*n+1) * (4*n+3) * 2^(4*n+3) ); pretty(a)

\[
\begin{array}{c|c|c}
N & a \\
\hline
0 & (-1) \\
1 & (-1) \\
2 & (-1) \\
3 & (-1) \\
\end{array}
\]

% echo off; diary off