Summary

In this section, we apply the theory and formulas developed in Section 10.7 to problems of a graphical nature. In this regard, we’ll dispense with hand work (mostly) and resort to unabashed use of machine firepower. Recall these pertinent concepts.

- **Taylor series**:
  \[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \quad \text{for } |x-a| < R. \]

- **Taylor polynomials**:
  \[ T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}, \quad n = 0, 1, 2, \ldots \]

- **Taylor’s formula for remainder** \( R_n(x) = f(x) - T_n(x) \):
  \[ R_n(x) = \frac{f^{(n+1)}(z)(x-a)^{n+1}}{(n+1)!}, \]
  where \( z \) is strictly between \( x \) and \( a \).

- **Taylor’s Inequality**: If \( |f^{(n+1)}(x)| \leq M \) for \( |x-a| < R \), then the remainder \( R_n(x) \) of the Taylor series satisfies
  \[ |R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \quad \text{for } |x-a| < R. \]

MATLAB Examples

MATLAB can compute Taylor polynomials with one command! Given a positive integer \( n \), \texttt{taylor(f,n)} returns the \( (n-1) \)th Taylor polynomial of \( f \) about 0, whereas \texttt{taylor(f,n,a)} centers the Taylor polynomial at \( a \). (The TI-89 also has a \texttt{taylor} command.)

\texttt{MATLAB} Example:

(a) Find the Taylor polynomials up to degree 3 for \( f(x) = 1/x \) centered at \( a = 1 \). Graph \( f \) and these polynomials on a common plot.

(b) Evaluate \( f \) and these polynomials at \( x = 0.9 \) and 1.3.

(c) Comment on how the Taylor polynomials converge to \( f(x) \).
Find \( T_4(x) \), the fourth degree Taylor polynomial of \( f(x) = \cos x \) centered at \( a = \frac{2}{3}\pi \). Graph \( f \) and \( T_4 \) on the same plot.

**Solution**

We have

\[
T_4(x) = -1/2 - \frac{1}{2} \sqrt{3} (x - 2/3 \pi) + \frac{1}{4} (x - 2/3 \pi)^2 + \frac{1}{12} \sqrt{3} (x - 2/3 \pi)^3 - \frac{1}{48} (x - 2/3 \pi)^4.
\]

Find the Taylor polynomials \( T_1, T_3, T_5, T_7, T_9 \) at \( a = 0 \) for \( f(x) = \tan x \). Graph these polynomials and \( f \) on the same screen.

**Solution**

You know the drill by now

\[
\begin{align*}
T_1(x) &= 1 \quad \text{for } T_1, \\
T_3(x) &= x + \frac{1}{6} x^3 \quad \text{for } T_3, \\
T_5(x) &= x + \frac{1}{12} x^5 \quad \text{for } T_5, \\
T_7(x) &= x + \frac{1}{24} x^7 \quad \text{for } T_7, \\
T_9(x) &= x + \frac{1}{120} x^9 \quad \text{for } T_9.
\end{align*}
\]
s644x12

(a) Approximate \( f(x) = \frac{1}{x} \) at \( a = 1 \) by \( T_3 \) for \( 0.8 \leq x \leq 1.2 \).

(b) Use Taylor’s Inequality to estimate the accuracy of the approximation \( f(x) \approx T_3(x) \) for \( 0.8 \leq x \leq 1.2 \).

(c) Check your result in (b) by graphing the remainder \( |R_3(x)| \).

Solution

(a) Here is a plot of \( f \) and \( T_3 \); nice fit!

(b) Since \( |f^{(3+1)}(x)| = \left| \frac{24}{x^5} \right| \leq \frac{24}{|0.8|^5} = 73.2422 = M \) for \( |x - 1| < 0.2 \), then the remainder \( R_3(x) \) of the Taylor series satisfies

\[
|R_3(x)| \leq \frac{M}{(3+1)!} |x - 1|^{3+1} \leq \frac{73.2422}{24} \cdot (0.2)^4 = 4.88 \times 10^{-3}
\]

for \( |x - 1| < 0.2 \). Thus \( U = 4.88 \times 10^{-3} \) is an upper bound on the absolute error.

(c) Here is a plot of \( |R_3(x)| \), the actual magnitude of the error in the approximation of \( f \) by \( T_3 \). Note that \( |R_3(x)| \leq U \) throughout the interval \((0.8, 1.2)\) as guaranteed by the theory!

Use the ASET or Taylor’s Inequality to estimate the range of values of \( x \) for which the approximation \( T_3(x) = 1 - \frac{1}{3}x^2 + \frac{1}{4}x^4 \) to \( f(x) = \cos x \) is accurate to within \( \epsilon = 0.005 \). Check your answer graphically.
Solution

The Maclaurin series for \( \cos x \) is
\[
1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 - \frac{1}{720} x^6 + \ldots
\]
Via the ASET, the error \( R_4(x) \) in the approximation \( T_4(x) \) satisfies
\[
|R_4(x)| \leq \frac{1}{720} x^6,
\]
the magnitude of the first neglected term of the series. Let’s see where this leads.

\[
|R_4(x)| \leq \frac{x^6}{720} \quad \text{want} \quad 0.005 = \frac{1}{200} \\
|x| \leq \frac{6\sqrt{720}}{200} \approx 1.24
\]

Therefore, the approximation \( T_4(x) = 1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 \) is within \( \epsilon = 0.005 \) of \( f(x) \) on the interval \((-1.24, 1.24)\). Here is a plot of \( |R_4(x)| \) over this interval that verifies this. Beauty, eh?

```matlab
% Stewart 644/22
% sym x
f = cos(x); % function
T = taylor(f, 5); % Taylor polynomial of degree 4
c = (720/200)^(1/6)
c = 1.2380
pretty(T)
x = linspace(-c, c);
abs_R = eval(vectorize(abs(f-T)));
plot(x, abs_R, 'r', 'LineWidth', 2); grid on
axis([-1.5 1.5 -1e-3 6e-3])
xlabel('x'); ylabel('| R_4(x) |')
title('Stewart 644/22: | R_4(x) |')
```

\[
x = \text{linspace}(-c, c);
abs_R = \text{eval(vectorize(abs(f-T)))};
\text{plot}(x, \text{abs}_R, \text{'r'}, \text{'LineWidth'}, 2); \text{grid on}
\text{axis}([-1.5 1.5 -1e-3 6e-3])
\text{xlabel}('x'); \text{ylabel}('| R_4(x) |')
\text{title}('Stewart 644/22: | R_4(x) |')
```