Summary

Looking ahead to Calc 3, we’ll briefly examine vector integrals, multiple integrals, and vector multiple integrals. While the limit-of-a-Riemann-sum approach is mentioned, in practice these types of integrals are evaluated in a mechanistic fashion. They are quite easy to do with a TI-89 calculator or in MATLAB. This is what we’ll usually do, especially in Calc 3.

Overview: Vector & Multiple Integrals

Spring 2005 Math 152
Thu, 03/Mar ☪2005, Art Belmonte

Review of Single Integrals from Calc 1

Definition Let \( f \) be a function defined on \( I = [a, b] \). Split \([a, b]\) into \( n \) subintervals whose endpoints constitute a partition

\[
P : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.
\]

(Often the \( x_i \) are equally spaced and we have a regular partition.) Let \( x_i^* \in [x_{i-1}, x_i] \) be in the \( i \)th subinterval and \( \Delta x_i = x_i - x_{i-1} \) be the length of this subinterval. The norm of \( P \) is defined by

\[
\|P\| = \max \Delta x_i.
\]

Now let the number of subintervals \( n \) increase indefinitely while the norm of \( P \) shrinks to 0. The definite integral of \( f \) from \( a \) to \( b \) is defined by

\[
\int_a^b f(x) \, dx = \lim_{\|P\| \to 0} \sum_{i=1}^n f(x_i^*) \Delta x_i,
\]

provided this limit of a Riemann sum exists. When this occurs, \( f \) is said to be integrable on \([a, b]\).

Antiderivatives The scalar function \( F(x) \) is an antiderivative or indefinite integral of the scalar function \( f(x) \) on an interval \( I \) if and only if \( F'(x) = f(x) \) for all \( x \in I \). A vector function \( \mathbf{R}(t) \) is an antiderivative or indefinite integral of the vector function \( \mathbf{r}(t) \) on an interval \( J \) if and only if we have \( \mathbf{R}'(t) = \mathbf{r}(t) \) for all \( t \in J \).

Fundamental Theorem of Calculus (FTC), Part 2 Computing a definite integral as a limit of a Riemann sum is quite tedious. But computing it with the FTC is easy. Let \( f \) be a continuous function on \([a, b]\). Then

\[
\int_a^b f(x) \, dx = F(x) \Big|_a^b = F(b) - F(a),
\]

where \( F \) is an antiderivative of \( f \).

Vector Integrals

Let \( \mathbf{r}(t) = [f(t), g(t)], \ a \leq t \leq b \), be a continuous vector function and \( P \) be a partition of \([a, b]\). The definite integral of \( \mathbf{r} \) from \( a \) to \( b \) is defined by

\[
\int_a^b \mathbf{r}(t) \, dt = \lim_{\|P\| \to 0} \sum_{i=1}^n \mathbf{r}(t_i^*) \Delta t_i
\]

In other words,

\[
\int_a^b f(t) \, dt = \int_a^b g(t) \, dt = \left[ \int_a^b f(t) \, dt, \int_a^b g(t) \, dt \right].
\]

We simply map the operation of integration onto the components of the vector function. Moreover, we may use the FTC to evaluate the component integrals.

Double Integrals

Definition Let \( f(x, y) \) be a continuous function over a rectangular region

\[
R = \{ (x, y) : a \leq x \leq b, c \leq y \leq d \}
\]

in the \( xy \)-plane. We define the double integral of \( f \) by “slicing and dicing,” as it were. (Think of mincing that onion with your Ginsu knife...) That is, split \([a, b]\) into \( m \) subintervals and \([c, d]\) into \( n \) subintervals. The norm \( \|P\| \) of the resulting partition \( P \) is the length of the longest diagonal among the subrectangles of the partition. We then form a double Riemann sum and take the limit as \( \|P\| \) shrinks to 0. If this limit exists, we obtain the double integral of \( f \) over \( R \).

\[
\iint_R f(x, y) \, dA = \lim_{\|P\| \to 0} \sum_{i=1}^m \sum_{j=1}^n f(x_i^*, y_j^*) \Delta x_i \Delta y_j
\]

Iterated Integrals Computing the exact value of a double integral by taking a limit of a double Riemann sum is very difficult even when it is possible.

A practical way to evaluate multiple integrals is via iterated integration, where we compute single integrals in succession.
In other words, we repeatedly compute antiderivatives and apply the Fundamental Theorem of Calculus, working from inside-out until we are finished.

This mechanistic approach has been fully automated, as in the smi (stepwise [multiple] integration) commands on the TI-89 and in MATLAB. Accordingly, this reduces the problem of computing multiple integrals to simply setting them up. That said, you ought to try some problems purely by hand to get some feeling for the work that you are being spared.

**Fubini’s Theorem** If \( f \) is continuous on a rectangular region 
\[ R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\} \]
then
\[ \int_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy. \]

**Nonrectangular regions**
- A Type I region has the form
  \[ D = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}. \]
  It is a region in the \( xy \)-plane bounded on the left by the vertical line \( x = a \), on the right by the vertical line \( x = b \), below by the curve \( y = g_1(x) \), and above by the curve \( y = g_2(x) \). The double integral of the function \( f(x, y) \) over \( D \) may be computed as
  \[ \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx. \]
- A Type II region has the form
  \[ D = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}. \]
  It is a region in the \( xy \)-plane bounded on the below by the horizontal line \( y = c \), above by the horizontal line \( y = d \), on the left by the curve \( x = h_1(y) \), and on the right by the curve \( x = h_2(y) \). The double integral of the function \( f(x, y) \) over \( D \) may be computed as
  \[ \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy. \]

**NOTES:** The aforementioned “curves” may be lines. Also, vertical or horizontal boundaries may actually collapse to a single point. Finally, some regions are of both types. Here are some samples of regions.

**Vector Double Integrals**

Let \( \mathbf{r}(x, y) = [f(x, y), g(x, y)] \) be a continuous function defined on a region \( D \) of Type 1 and/or Type 2. The vector double integral of \( \mathbf{r} \) over \( D \) is just what you’d think.

\[
\begin{align*}
\int_D \mathbf{r}(x, y) \, dA &= \int_D [f(x, y), g(x, y)] \, dA \\
&= \left[ \int_D f(x, y) \, dA, \int_D g(x, y) \, dA \right]
\end{align*}
\]

Just map the integration operation onto the components of the vector function. Moreover, we may use Fubini’s Theorem on general regions to evaluate the component integrals.

**Generalizations**

Let’s extend the aforementioned concepts.

- If our vector function has three components, then
  \[ \int_D \mathbf{r}(t) \, dt = \left[ \int_D f(t) \, dt, \int_D g(t) \, dt, \int_D h(t) \, dt \right]. \]
- If \( f(x, y, z) \) is continuous on the rectangular box
  \[ R = \{(x, y, z) : a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\} \]
then the triple integral of \( f \) over \( R \) is defined by
  \[
  \iiint_R f(x, y, z) \, dV = \int_r^s \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz \\
  = \lim_{\|P\| \to 0} \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^s f(x_i^*, y_j^*, z_k^*) \Delta x_i \Delta y_j \Delta z_k.
  \]

Similarly, Fubini’s Theorem may be extended to allow computation of a triple integral via iterated integration. Moreover, triple integrals over more general regions \( E \) may be realized.

\[
\begin{align*}
\iiint_E f(x, y, z) \, dV &= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(y)}^{h_2(y)} f(x, y, z) \, dz \, dy \, dx \\
&= \left[ \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(y)}^{h_2(y)} f(x, y, z) \, dz \, dy \, dx \right]
\end{align*}
\]

- Let \( \mathbf{r}(x, y, z) = [f(x, y, z), g(x, y, z), h(x, y, z)] \) be a continuous function defined on a suitable region \( E \) in 3-D \( xyz \)-space. The vector triple integral \( \iiint_E \mathbf{r}(x, y, z) \, dV \) of \( \mathbf{r} \) over \( E \) is
  \[
  \begin{align*}
  \iiint_E [f(x, y, z), g(x, y, z), h(x, y, z)] \, dV &= \left[ \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(y)}^{h_2(y)} f(x, y, z) \, dz \, dy \, dx \right] \\
  &= \left[ \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(y)}^{h_2(y)} g(x, y, z) \, dz \, dy \, dx \right] \\
  &= \left[ \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(y)}^{h_2(y)} h(x, y, z) \, dz \, dy \, dx \right]
  \end{align*}
\]

Just map the integration operation onto the components of the vector function and use the Fubini’s Theorem to evaluate the component integrals.