vector function: a mapping \( r : D \rightarrow \mathbb{R}^n \) from a subset of the real line to \( n \)-dimensional real space: \( r : t \mapsto r(t) \). (Typically \( n = 2 \) for the \( xy \)-plane and \( n = 3 \) for \( xyz \)-space.) The input parameter is often denoted by \( t \) (for time), whereas the vector \( r(t) \) is a position vector.

operations on vector functions: Simply map the given operation onto the components of the vector function. For example, the limit / derivative / integral of a vector function is just the vector of limits / derivatives / integrals of the component functions.

space curve: graph of a vector function in space.

Derivative rules involving vector functions

In the following \( u \) and \( v \) are vector functions, \( f \) is a scalar function, and \( k \) is a constant. The prime (’) mark signifies differentiation with respect to \( t \).

1. \( \frac{d}{dt} (u(t) + v(t)) = u'(t) + v'(t) \)
2. \( \frac{d}{dt} (kv(t)) = kv'(t) \)
3. \( \frac{d}{dt} (f(t)v(t)) = f'(t)v(t) + f(t)v'(t) \)
4. \( \frac{d}{dt} (u(t) \cdot v(t)) = u'(t) \cdot v(t) + u(t) \cdot v'(t) \)
5. \( \frac{d}{dt} (u(t) \times v(t)) = u'(t) \times v(t) \times u(t) \times v'(t) \)
6. \( \frac{d}{dt} (v(f(t))) = f'(t)v'(t) \), via the Chain Rule

Hand Examples

697/26
Find the limit \( \lim_{t \to \infty} \left( e^{-t}i + \frac{t-1}{t+1} j + \tan^{-1} t k \right) \).

Solution
The limit of the vector is the vector of limits.
\[
\left[ \lim_{t \to \infty} e^{-t}, \lim_{t \to \infty} \frac{t-1}{t+1}, \lim_{t \to \infty} \tan^{-1} t \right] = \left[ 0, 1, \frac{\pi}{2} \right]
\]

697/28
Find the domain of the vector function
\[
r(t) = \left[ t^2 - 4, \sqrt{t - 4}, \sqrt{6 - t} \right]
\]
and its derivative.

Solution
- The derivative of the vector is the vector of derivatives.
We have \( r'(t) = \left[ 2t, \frac{1}{2} (t - 4)^{-1/2}, -\frac{1}{2} (6 - t)^{-1/2} \right] \)
or \( \left[ 2t, \frac{1}{2\sqrt{t - 4}}, -\frac{1}{2\sqrt{6 - t}} \right] \).
- The domain of \( r \) is \( 4 \leq t \leq 6 \), whereas that of \( r' \) is \( 4 < t < 6 \).

697/42
Find the unit tangent vector of \( r(t) = [e^t, e^{-2t}, te^{2t}] \) at \( t = 0 \).

Solution
- The tangent vector is
\[
v = r'(t) = \left[ 2e^t, -2e^{-2t}, (2t+1)e^{2t} \right] = [2, -2, 1]
\]
when \( t = 0 \).
- Hence the unit tangent vector at \( t = 0 \) is
\[
\hat{v} = \frac{v}{\|v\|} = \frac{[2, -2, 1]}{\sqrt{4 + 4 + 1}} = \left[ \frac{2}{3}, -\frac{2}{3}, 1 \right].
\]

698/62
Find \( r(t) \) if \( r'(t) = [\sin t, -\cos t, 2t] \) and \( r(0) = [1, 1, 2] \).

Solution
Antidifferentiation yields \( r(t) = [-\cos t, -\sin t, t^2] + C \). Thus
\[
[1, 1, 2] = r(0) = [-1, 0, 0] + C.
\]
Hence \( C = [1, 1, 2] - [-1, 0, 0] = [2, 1, 2] \). Therefore,
\[
r(t) = \left[ 2 - \cos t, -1 - \sin t, t^2 + 2 \right]
\]
**MATLAB Examples**

**s696x10**

Sketch the space curve for the vector function
\[ \mathbf{r}(t) = [\sin t, t, \cos t]. \]

Indicate with arrows the direction in which \( t \) increases.

**Solution**

The space curve is a *helix*, like the cord on a land-line telephone. Here is an M-file followed by a graph.

```matlab
% delete s696x10.txt; diary s696x10.txt
clear; clc; close all; echo on
% Stewart 696/10
% t = linspace(0, 12*pi, 500);
x = sin(t); y = t; z = cos(t);
plot3(x,y,z); axis equal
% echo off; diary off
```

**s697x38**

(a) Sketch the plane curve for the vector function
\[ \mathbf{r}(t) = (2 \sin t) \mathbf{i} + (3 \cos t) \mathbf{j}. \]

(b) Find the derivative \( \mathbf{r}'(t) \).

(c) Sketch the position vector \( \mathbf{r}(t) \) and the tangent vector \( \mathbf{r}'(t) \) for \( t = \frac{\pi}{4} \).

**Solution**

We have \( \mathbf{r}'(t) = [2 \cos t, -3 \sin t] = \left[ 1, -\frac{3\sqrt{2}}{2} \right] \) at \( t = \frac{\pi}{4} \).

```matlab
% Stewart 697/38
% sym t
r = [2*sin(t), 3*cos(t)]; v = diff(r,t)
v = [ 2*cos(t), -3*sin(t)]
```

**s697x50**

Find parametric equations for the tangent line to the curve
\[ \mathbf{r}(t) = [\cos t, 3e^{2t}, 3e^{-2t}] \] at the point \( A(1, 3, 3) \). Illustrate by graphing the curve and tangent line on the same figure.

**Solution**

Note that point \( A \) corresponds to parameter value \( t = 0 \). (For \( 3e^{2t} = 3 \) implies \( e^{2t} = 1 \), whence \( 2t = \ln 1 = 0 \) and thus \( t = 0 \).) A direction vector for the tangent line is the tangent vector at \( A \):
\[ \mathbf{v} = \mathbf{r}'(0) = [-\sin t, 6e^{2t}, -6e^{-2t}] \] at \( t = 0 \). Thus an equation of this line is
\[ \mathbf{L}(u) = \mathbf{A} + u\mathbf{v} = [1, 3 + 6u, 3 - 6u]. \]

```matlab
% Stewart 697/50
% sym t
r = [cos(t), 3*exp(2*t), 3*exp(-2*t)];
v = diff(r,t)
v = [ -sin(t), 6*exp(2*t), -6*exp(-2*t)]
t0 = sym(0);
r0 = simple(subs(r, t, t0))
```
At what point do the curves \( r_1(t) = [t, 1-t, 3+t^2] \) and 
\( r_2(s) = [3-s, s-2, s^2] \) intersect?

**Solution**

Equate the vectors, simultaneously solve for \( s \) and \( t \), then substitute.

\[
\begin{align*}
\text{at} & = \begin{bmatrix} t-3+s, \\ 3-t-s, \\ 3+t^2-s^2 \end{bmatrix} \\
\text{in} & = \text{solve}(q(1), q(2), q(3)) \\
\text{at} & = 2 \\
\text{in} & = 1
\end{align*}
\]
\[ L_1 = \begin{bmatrix} 1, & 2, & -u\pi \end{bmatrix} \]

\[ q = L_0 - L_1 \]

\[ q = \begin{bmatrix} \pi t - 1, & 2\pi t - 2, & 1 + u\pi \end{bmatrix} \]

\[ s = \text{solve}(q(1), q(2), q(3)) \]

Warning: 3 equations in 2 variables.

\[ s = \begin{bmatrix} u \end{bmatrix}, \begin{bmatrix} t \end{bmatrix} \]

\[ t = s.t \]

\[ u = s.u \]

\[ P = \text{subs}(L_0) \]

\[ \begin{bmatrix} 1, & 2, & 1 \end{bmatrix} \]

\[ \text{check} = \text{subs}(L_1) \]

\[ \begin{bmatrix} 1, & 2, & 1 \end{bmatrix} \]

\[ \int_1^2 \begin{bmatrix} 1 + t^2, & -4t^4, & 1 - t^2 \end{bmatrix} \, dt \]

Solution

The integral of the vector is the vector of integrals.

\[ \int_1^2 \begin{bmatrix} 1 + t^2, & -4t^4, & 1 - t^2 \end{bmatrix} \, dt = \begin{bmatrix} \int_1^2 (1 + t^2) \, dt, & \int_1^2 -4t^4 \, dt, & \int_1^2 (1 - t^2) \, dt \end{bmatrix} = \begin{bmatrix} \frac{10}{3}, & -\frac{124}{5}, & -\frac{4}{3} \end{bmatrix}. \]