Spring 2004  Math 253/501–503  
11 Three Dimensional Analytic Geometry and Vectors  
11.7 Arc Length and Curvature  
Tue, 27/Jan ©2004, Art Belmonte

Summary

Let \( \mathbf{r} \) be an \( n \)-dimensional vector function, unless otherwise noted.

- The **arc length** of a curve from \( t = a \) to \( t = b \) is
  \[
  L = \int_a^b \| \mathbf{r}'(t) \| \, dt = \int_a^b \sqrt{ \sum_{k=1}^n \left( \frac{d}{dt} r_k(t) \right)^2 } \, dt.
  \]

- The **arc length function** is \( s = s(t) = \int_a^t \| \mathbf{r}'(u) \| \, du \).

- **unit tangent vector**: \( \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\| \mathbf{r}'(t) \|} \)

- **unit normal vector**: \( \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\| \mathbf{T}'(t) \|} \)

- **unit binormal vector**: \( \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \), in 3-D space

- The **curvature** \( \kappa \) of a space curve \( \mathbf{r}(t) \) measures how quickly the curve changes direction.
  \[
  \kappa = \sqrt{ \frac{ \| \mathbf{T}'(t) \| }{ \| \mathbf{r}'(t) \| } } = \frac{ \| \mathbf{r}'(t) \times \mathbf{r}''(t) \| }{ \| \mathbf{r}'(t) \|^3 }.
  \]

- In the case of a plane curve \( y = f(x) \), we have an alternative formula for curvature.
  \[
  \kappa = \frac{ f''(x) }{ \left( 1 + (f'(x))^2 \right)^{3/2} }.
  \]
  Or just use the preceding general formula.

- The **torsion** \( \tau \) measures the degree of twisting of a curve.
  \[
  \tau = \frac{ (\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}''''(t) }{ \| \mathbf{r}'(t) \times \mathbf{r}''(t) \|^2 }.
  \]

Hand Examples

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Find the arc length of the space curve
\[
\mathbf{r}(t) = [e^t, e^t \sin t, e^t \cos t], \quad 0 \leq t \leq 2\pi.
\]

Solution

Now \( \mathbf{r}'(t) = [e^t, e^t (\cos t + \sin t), e^t (\cos t - \sin t)] \), whence
\[
\| \mathbf{r}'(t) \| = \sqrt{ e^{2t} + e^{2t} (1 + 2 \cos t \sin t) + e^{2t} (1 - 2 \cos t \sin t) } = \sqrt{e^{2t}}. 
\]
Now use the arc length formula.
\[
L = \int_a^b \| \mathbf{r}'(t) \| \, dt \\
= \int_0^{2\pi} \sqrt{e^{2t}} \, dt \\
= \sqrt{e^{2\pi}} - 1 \approx 925.77
\]

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Reparameterize the curve \( \mathbf{r}(t) = (1 + 2t) \mathbf{i} + (3 + t) \mathbf{j} - 5t \mathbf{k} \) from the point where \( t = 0 \) in the direction of increasing \( t \).

Solution

Use the arc length function.
\[
s = s(t) = \int_a^t \| \mathbf{r}'(u) \| \, du \\
= \int_0^t \sqrt{4 + 1 + 25} \, du \\
= \sqrt{30} t.
\]
Thus \( s = \sqrt{30} t \), whence \( t = t(s) = \frac{1}{\sqrt{30}} s \). Therefore,
\[
\mathbf{r}(s) = \mathbf{r}(t(s)) = [1 + \frac{2s}{\sqrt{30}}, 3 + \frac{s}{\sqrt{30}}, -5s].
\]

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Find the space curve \( \mathbf{r}(t) = [\sqrt{3} \cos t, \sin t, \sin t] \), compute the unit tangent vector, the unit normal vector, and the curvature.

Solution

- The tangent vector is \( \mathbf{r}'(t) = [-\sqrt{3} \sin t, \cos t, \cos t] \). Hence the unit tangent vector is
  \[
  \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\| \mathbf{r}'(t) \|} = \frac{[-\sqrt{3} \sin t, \cos t, \cos t]}{\sqrt{2 \sin^2 t + \cos^2 t + \cos^2 t}} \\
  = \frac{1}{\sqrt{2}} \left[ -\sqrt{3} \sin t, \cos t, \cos t \right] \\
  = \left[ -\sin t, \frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \cos t \right].
  \]
Now $\mathbf{T}'(t) = \left[ -\cos t, -\frac{1}{\sqrt{2}} \sin t, -\frac{1}{\sqrt{2}} \sin t \right]$, a unit vector. Hence the unit normal vector is $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \mathbf{T}'(t) = \left[ -\cos t, -\frac{1}{\sqrt{2}} \sin t, -\frac{1}{\sqrt{2}} \sin t \right]$. The curvature is $\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{T}'(t)\|} = \frac{1}{\sqrt{2}}$.

The torsion is given by $\tau = \frac{\mathbf{T}'(t) \times \mathbf{T}''(t) \cdot \mathbf{T}'''(t)}{\|\mathbf{T}'(t) \times \mathbf{T}''(t)\|^2}$.

MATLAB Examples

s705x06

Use a computer to graph the curve with parametric equations

$$x = \cos t, \quad y = \sin 3t, \quad z = \sin t.$$ 

for $0 \leq t \leq 2\pi$. Then numerically approximate its arc length.
Find the curvature of the curve \( y = \ln x \).

Solution

Embed the plane curve in space, then use \( \kappa \).

\[
s705x24
\]

\[
\frac{1}{2} \quad 6
\]
\[
\frac{1}{2} \quad \frac{2}{3} \quad (2t - 4t + 5)
\]

At what point does the curve \( y = e^x \) have maximum curvature?

Solution

Set the derivative of the curvature to zero and solve. Verify with the second derivative test.

\[
s706x47 [706/47 revisited]
\]

Find the torsion of the space curve \( r = [t, \frac{1}{2}t^2, \frac{1}{3}t^3] \).

Solution

Use the \( \tau \) command I wrote.

\[
s705x25
\]