4.25
12.25
8.25
36x459
norm
size of the subintervals (the
and \([a, b]\) into \(m\) subintervals (typically of equal
length) whose endpoints constitute a partition

\[ P : a = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = b. \]

Let \(x_i^*\) be a point in the \(i\)th subinterval, \([x_{i-1}, x_i]\). Now let the
number of subintervals increase indefinitely, while the maximum
size of the subintervals (the norm of \(P\)) shrinks to 0. If the limit

\[ \lim_{\|P\| \to 0} \sum_{i=1}^{m} f(x_i^*) \Delta x_i \]

exists, we call it the definite integral of \(f\)
from \(a\) to \(b\), written \(\int_{a}^{b} f(x) \, dx\), and say that \(f\) is integrable.

\[ \int_{a}^{b} f(x) \, dx = \lim_{\|P\| \to 0} \sum_{i=1}^{m} f(x_i^*) \Delta x_i \]

Double Riemann Integral

In Calc 3, we similarly define the double integral of a function
\(f(x, y)\) over a rectangular region

\[ R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\} \]

by “slicing and dicing,” as it were. (Think of mincing that onion
with your Ginsu knife...) That is, split \([a, b]\) into \(m\) subintervals
and \([c, d]\) into \(n\) subintervals. (Typically, the \(x\)-intervals are
equal-length, as are the \(y\)-subintervals.) The norm \(\|P\|\) of the
resulting partition \(P\) is the length of the longest diagonal among
the subrectangles of the partition. We then form a double Riemann
sum and take the limit as \(\|P\|\) shrinks to 0. If this limit exists, we
obtain the double integral of \(f\) over \(R\).

\[ \iint_{R} f(x, y) \, dA = \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx = \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy \]

\[ = \lim_{\|P\| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta x_i \Delta y_j \]

Remarks

We’ll actually compute a double integral exactly in this handout
by having MATLAB take the limit of a double Riemann sum.
Since this is rather difficult to do by hand, we’ll often merely use a
particular double Riemann sum as an approximation to a double
integral.

In the next section, we’ll learn how to compute double integrals
exactly the easy way by repeated application of the Fundamental
Theorem of Calculus (antidifferentiation followed endpoint
evaluation of the antiderivative and subtraction).

Note, however, that the approximate methods we learn in this
section are still useful—both as an extra check on our work and in
cases where antidifferentiation is impossible.

In addition to doing some exercises by hand, we shall employ our
TI-89s and MATLAB to fully automate computations!

“Hand” Examples

We continue to utilize machine power. Use your TI-89 and
TAMUCALC when doing problems “by hand.”

799/4

Calculate the double Riemann sum of \(f(x, y) = x^2 + 4y\) for the
partition of \(R = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 3\}\) determined by
the lines \(x = 1, y = 1, y = 2\). Compute function values of \(f\) at
the centers of the subrectangles.

Solution

- The line \(x = 1\) partitions the \(x\)-interval into two \((m = 2)\)
  subintervals of equal length, \(h = \Delta x = \frac{2-0}{m} = \frac{2}{2} = 1\).
- The lines \(y = 1, 2\) partition the \(y\)-interval into three \((n = 3)\)
  subintervals of equal length, \(k = \Delta y = \frac{3-0}{n} = \frac{3}{3} = 1\).
- Centers of the subrectangles determined by the partition are:
  \[
  \begin{pmatrix}
  1 & 5/2 \\
  1/2 & 5/2 \\
  1 & 5/2
  \end{pmatrix}
  \begin{pmatrix}
  3/2 & 5/2 \\
  3/2 & 5/2 \\
  3/2 & 5/2
  \end{pmatrix}
  \]
  \[
  \begin{pmatrix}
  1 & 3/2 \\
  1/2 & 3/2 \\
  1 & 3/2
  \end{pmatrix}
  \begin{pmatrix}
  3/2 & 3/2 \\
  3/2 & 3/2 \\
  3/2 & 3/2
  \end{pmatrix}
  \]
- The function values of \(f\) at these points are:
  \[
  \begin{pmatrix}
  10.25 & 12.25 \\
  6.25 & 8.25 \\
  2.25 & 4.25
  \end{pmatrix}
  \]
Accordingly, the double Riemann sum is
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} f \left( x_i, y_j \right) \Delta x_i \Delta y_j = \Delta x \Delta y \text{(sum of } f\text{-values)}
\]
\[
= (1)(1) \left( 18 + \frac{6}{3} \right)
\]
\[
= 43.5
\]

Clearly this is a lot of computation to do by hand! When the subintervals of the partition are fixed-length, this entire procedure may be fully automated. First define \( f \) via the T A M U C A L C command \( mf \). Then use the command \( \text{dblsummd}"f", 0, 2, 2, 0, 3, 3 \) to obtain 43.5, as before. (See the supplemental T A M U C A L C handout for details.)

799/6

Calculate the double Riemann sum of \( f(x, y) = 2x + x^2y \) for the partition of \( R = \{(x, y): -2 \leq x \leq 2, -1 \leq y \leq 1\} \) determined by the lines \( x = -1, x = 0, x = 1 \) and \( y = -\frac{1}{2}, y = 0, y = \frac{1}{2} \).

Solution

“Use the Force, Luke.” Define \( f \) via \( mf \). Then, for the stated partition, we have \( a = -2 \), \( b = 2 \), \( m = 4 \), \( c = -1 \), \( d = 1 \), \( n = 4 \) Therefore, \( \text{dblsumbl}"f", -2, 2, 4, -1, 1, 4 \) yields \(-11 \).

799/16

If \( R = [0, 1] \times [0, 1] \), show that \( 0 \leq \iint_R \sin (x + y) \ dA \leq 1 \).

Solution

The notation \( [0, 1] \times [0, 1] \) signifies the rectangular region \( R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\} \). On this region, we have \( 0 \leq x + y \leq 2 \), whence \( 0 \leq \sin (x + y) \leq 1 \). Thus
\[
0 = \iint_R 0 \ dA \leq \iint_R \sin (x + y) \ dA \leq \iint_R 1 \ dA = \text{area of } R = 1
\]
That is, \( 0 \leq \iint_R \sin (x + y) \ dA \leq 1 \).

MATLAB Examples

s799x04 [799/4 revisited]

Calculate the double Riemann sum of \( f(x, y) = x^2 + 4y \) for the partition of \( R = \{(x, y): 0 \leq x \leq 2, 0 \leq y \leq 3\} \) determined by the lines \( x = 1, y = 1, y = 2 \).

Solution

Here we replicate the work we did with our TI-89. Cooper’s \texttt{riemann} routine computes a double Riemann sum using equal-length subintervals, with function values computed at the centers of subrectangles. It also provides a nice graphical visualization of the volume under a surface! The “height” at a point on the surface is approximated by a piecewise-constant function; sort of like a city-scape.

% Stewart 799/4
% f = inline('x.^2 + 4*y', 'x', 'y');
corners = [0 2 0 3];
riemann(f, corners, 1)
enter the number of subdivisions in x and y direction as [n m]: [2 3]
subdiv = 2 3
Approximate value of the integral ans = 43.5000
% echo off; diary off

MATLAB Examples

s799x04 [799/4 revisited]

For \( m = 2 \) and \( n = 3 \), the double Riemann is 43.5. When \( m = 6 \) and \( n = 9 \), it is 43.94. As \( m \to \infty \) and \( n \to \infty \), the limit of the Riemann sums is 44.
Calculate the double integral \( \int_0^3 \int_0^2 x^2 + 4y \, dx \, dy \) exactly by taking the limit of a symbolic Riemann sum.

**Solution**

We split \([0, 2]\) into \(m\) equal-length subintervals and \([0, 3]\) into \(m\) equal-length subintervals. We then compute the symbolic Riemann sum \( \sum_{i=1}^m \sum_{j=1}^m f(x_i^*, y_j^*) \Delta x_i \Delta y_j \) using the centers of subrectangles. Finally, we let \(m \to \infty\) to obtain 44, the exact value of the double integral.

```matlab
syms ii jj m x y
f = x^2 + 4*y;
a = 0; b = 2; c = 0; d = 3;
h = (b-a)/m, k = (d-c)/m
h = 2/m
k = 3/m
xi = a + (ii - 1/2)*h, yj = c + (jj - 1/2)*k
xi = 2*(ii-1/2)/m
yj = 3*(jj-1/2)/m
fij = subs(f, [x y], [xi yj])

ss = h * k * symsum(symsum(fij, ii,1,m), jj,1,m);
pretty(ss)  # numerical result

definite_integral = limit(ss, m, inf)
definite_integral = 44
```

**TAMU CALC double Riemann sum examples**

Here are the five double Riemann sum approximations to \( \int_{3/4}^1 \int_0^{1/2} e^{x+y} \, dx \, dy \) given in the TAMU CALC Command Reference. Here we use the MATLAB routine `drs` (double Riemann sum) that I wrote this weekend. Type `help drs` in the MATLAB Command Window for details on the usage of `drs`.

```matlab
f = inline('exp(x+y)', 'x', 'y');
a = zeros(5,1);
a(1) = drs(f, [0 1/2 4; 3/4 1 8], 'bl');
a(2) = drs(f, [0 1/2 4; 3/4 1 8], 'br');
a(3) = drs(f, [0 1/2 4; 3/4 1 8], 'tl');
a(4) = drs(f, [0 1/2 4; 3/4 1 8], 'tr');
a(5) = drs(f, [0 1/2 4; 3/4 1 8], 'md');
a
a =
0.3605
0.4085
0.3719
0.4215
0.3898
```

**Solution**

The number of squares is \(m \times n = 2^k \times 2^k, k = 0, 1, 2, 3, 4, 5\). Here is the needful. The `drs` routine is discussed below.

```matlab
f = inline('cos(x.^4 + y.^4)'; 'x', 'y');
a = zeros(6,1);
for k = 0:5
    a(k+1) = drs(f, [0 1 2^k; 0 1 2^k], 'md');
end
```

**Use MATLAB to estimate** \( \int_R \cos(x^4 + y^4) \, dA \), where \( R = [0, 1] \times [0, 1] \). Use the Midpoint Rule with the following numbers of squares of equal size: 1, 4, 16, 64, 256, and 1024.

```matlab
% TAMU CALC double Riemann sum examples
f = inline('exp(x+y)', 'x', 'y');
a = zeros(5,1);
a(1) = drs(f, [0 1/2 4; 3/4 1 8], 'bl');
a(2) = drs(f, [0 1/2 4; 3/4 1 8], 'br');
a(3) = drs(f, [0 1/2 4; 3/4 1 8], 'tl');
a(4) = drs(f, [0 1/2 4; 3/4 1 8], 'tr');
a(5) = drs(f, [0 1/2 4; 3/4 1 8], 'md');
a
a =
0.3605
0.4085
0.3719
0.4215
0.3898
```

**Use MATLAB to estimate** \( \int_R \cos(x^4 + y^4) \, dA \), where \( R = [0, 1] \times [0, 1] \). Use the Midpoint Rule with the following numbers of squares of equal size: 1, 4, 16, 64, 256, and 1024.

```matlab
% TAMU CALC double Riemann sum examples
```

**Solution**

The number of squares is \(m \times n = 2^k \times 2^k, k = 0, 1, 2, 3, 4, 5\). Here is the needful. The `drs` routine is discussed below.

```matlab
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a = zeros(6,1);
for k = 0:5
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end
```

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end
```

**Use MATLAB to estimate** \( \int_R \cos(x^4 + y^4) \, dA \), where \( R = [0, 1] \times [0, 1] \). Use the Midpoint Rule with the following numbers of squares of equal size: 1, 4, 16, 64, 256, and 1024.

```matlab
% TAMU CALC double Riemann sum examples
```