Summary

Recall the relationships between rectangular \((x, y, z)\), cylindrical \((r, \theta, z)\), and spherical \((\rho, \theta, \phi)\) coordinates.

- Cylindrical / rectangular relationships
  \[x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \quad z = z\]
  Note that (typically) \(r \geq 0\) and \(0 \leq \theta \leq 2\pi\).

- Spherical / rectangular relationships
  \[x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi \]
  \[r = \rho \sin \phi \quad \rho^2 = x^2 + y^2 + z^2 \]
  Note that \(\rho \geq 0\), \(0 \leq \phi \leq \pi\), and \(0 \leq \theta \leq 2\pi\).

- Here are three specifications of the volume differential \(dV\): rectangular, cylindrical, and spherical, respectively.
  \[dx \, dy \, dz = r \, d\rho \, d\theta \, d\phi = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta\]

When we cover Jacobian matrices and determinants in Section 13.11, we’ll see how the extra factors \(r\) and \(\rho^2 \sin \phi\) are obtained. Until then, just look at the figures 3 and 7 in Section 13.10 of Stewart for a geometrical explanation.

Hand / MATLAB Examples

Aids to our hand work are the TAMUCALC MuInt menu and the smi command. The former contains a variety of multiple integral templates that facilitate typing. The latter shows the steps involved in integration.

Pay careful attention to the 2-D and 3-D graphics in the examples. These will help you to set up the limits of integration in your triple integrals. With practice, you’ll be able to draw rough sketches that will suffice. In upcoming computer assignments, you will draw some graphs yourselves with MATLAB.

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Sketch the solid whose volume is given by the integral
\[\int_0^{2\pi} \int_0^{\pi/3} \int_0^{\sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta,\]
then compute this volume.

Solution

The \(\theta\) - and \(\phi\)-specifications indicate that the solid lies inside a cone. For fixed \(\theta\) and \(\phi\), the innermost variable \(\rho\) varies from \(\rho = 0\) (the origin) to \(\rho = \sec \phi\) or \(z = \rho \cos \phi = 1\), a horizontal plane. What we have here is an HCFT: a half-cone with a flat top! The volume is
\[V = \int \int \int_E 1 \, dV = \int_0^{2\pi} \int_0^{\pi/3} \int_0^{\sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \pi \approx 3.14 \text{ cm}^3.\]

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Evaluate \(\int \int \int_E \sqrt{x^2 + y^2} \, dV\), where \(E\) is the solid bounded by the paraboloid \(z = 9 - x^2 - y^2\) and the \(xy\)-plane.

Solution

Here is a picture of the solid.
Switch to cylindrical coordinates. The paraboloid intersects the $xy$-plane in the circle $r^2 = x^2 + y^2 = 9$, $z = 0$. Hence the value of the integral is

$$\int_0^{2\pi} \int_0^3 \int_0^{9-r^2} r \cdot r \, dz \, dr \, d\theta = \frac{324}{5} \pi \approx 203.575.$$

% Stewart 853/6
% sym r t z
volume = int(int(int(r * r, ..., z,0,9-rˆ2), r,0,3), t,0,2*pi);
pretty(volume)

floated = eval(volume)
floated = 203.5752
% echo off; diary off

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Evaluate $\iiint_E x e^{(x^2+y^2+z^2)} \, dV$, where $E$ is the solid that lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.

**Solution**

Here is a picture, with a transparent outer sphere through which we see the inner sphere.

Switch to spherical coordinates, then evaluate the integral.

$$\int_{0}^{\pi/2} \int_{0}^{\pi/2} \int_{1}^{2} \rho \sin \phi \cos \theta \cdot e^{\rho^4} \cdot \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

$$= \frac{\pi}{16} \left(e^{16} - e\right) \approx 1.745 \times 10^6$$

% Stewart 854/18
% sym r t
value = int(int(int(r*sin(p)*cos(t) * exp(rˆ4) * rˆ2*sin(p), ..., r,1,2), p,0,pi/2), t,0,pi/2);
pretty(factor(value))

floated = eval(value)
floated = 1.7448e+06
% echo off; diary off

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Find the volume of the solid that lies above the cone $\phi = \frac{\pi}{2}$ and below the offset sphere $\rho = 4 \cos \phi$.

**Solution**

Here is a picture of the solid.
The volume is

\[ V = \iiint_E 1 \, dV \]

\[ = \int_0^{2\pi} \int_0^{\pi/3} \int_0^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]

\[ = 10\pi \approx 31.4 \text{ cm}^3. \]

Switching to cylindrical coordinates we have

\[ \int_0^{\pi/2} \int_0^1 \int_0^r r \cos \theta \cdot r \sin \theta \cdot z \cdot r \, dz \, dr \, d\theta = \frac{1}{96} \approx 0.0104. \]

Evaluate the integral

\[ \int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2 + y^2 + z^2 \, dz \, dx \, dy \]

by changing to spherical coordinates.

Solution

The region of integration lies in the first octant, with the cone \( z = \sqrt{x^2 + y^2} \) below and the sphere \( x^2 + y^2 + z^2 = 18 \) above.

Here is a picture showing boundary surfaces of the solid.

The cone is \( \rho \cos \phi = z = \sqrt{x^2 + y^2} = r = \rho \sin \phi \), whence \( \tan \phi = 1 \) or \( \phi = \frac{\pi}{4} \). Switching to spherical coordinates we have

\[ \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{18}} \rho^2 \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{486\pi}{5} (\sqrt{2} - 1) \approx 126.485. \]
Show that
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{x^2 + y^2 + z^2} e^{-x^2-y^2-z^2} \, dx \, dy \, dz = 2\pi.
\]

Solution

The region of integration is all of 3-D space. So integrate over a spherical ball of radius \(a\) centered at the origin, then take a limit of the result as \(a \to \infty\).

\[
\lim_{a \to \infty} \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{a} e^{-\rho^2} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \lim_{a \to \infty} \frac{2\pi \left( e^{a^2} - a^2 - 1 \right)}{a^2} = 2\pi
\]

% Stewart 854/38
% syms a positive
syms p r t
sphere_value = int(int(int( r*exp(-r^2) * r^2*sin(p), ... 
  r,0,a), p,0,pi), t,0,2*pi);
pretty(sphere_value)

\[
\frac{1}{2} (a^2 + 1 - \exp(a^2)) \pi
\]

limiting_value = limit(sphere_value, a, inf);
pretty(limiting_value)

\[
2 \pi
\]

% echo off; diary off