Summary

- Polar / rectangular relationships
  \[ x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \]

- If \( r > 0 \), we march forward along the ray \( \theta = \alpha \). If \( r < 0 \), we march backward along said ray.

- If \( \theta > 0 \), the angle is measured \textit{counterclockwise} from the positive \( x \)-axis (the polar axis). If \( \theta < 0 \), the angle is measured \textit{clockwise} from the positive \( x \)-axis. (This is the way the ancient Egyptians did it and this is the way we do it.)

- For a polar curve \( r = f(\theta) \), form \( x = r \cos \theta = f(\theta) \cos \theta \) and \( y = r \sin \theta = f(\theta) \sin \theta \), then compute \( \frac{dy}{dx} \) parametrically as
  \[ \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}. \]

- The arc length along a polar curve \( r = f(\theta) \) from \( \theta = \alpha \) to \( \theta = \beta \) is
  \[ L = \int_{\alpha}^{\beta} \sqrt{\left( \frac{dx}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2} \, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta. \]

Hand Examples

Coordinate/angle conversion commands are at the bottom of the TAMUCALC Vect menu. The \texttt{autopolr} command is near the top of the TAMUCALC FVMD menu.

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Plot the point with polar coordinates \( (r, \theta) = (2, -\pi/7) \). Then find two other pairs of polar coordinates of this point, one with \( r > 0 \) and one with \( r < 0 \).

Solution

Here’s a plot of the point drawn with MATLAB’s \texttt{polar} command.

Angles around the outermost circle are measured in degrees counterclockwise. The concentric circles depict varying radii \( r \). Another pair of polar coordinates for this point with \( r > 0 \) is \( (2, -\frac{7\pi}{2} + 2\pi) = (2, \frac{13\pi}{2}) \). A different pair of polar coordinates for this point with \( r < 0 \) is \( (-2, -\frac{7\pi}{2} + \pi) = (-2, \frac{5\pi}{2}) \).

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Plot the point with polar coordinates \( (r, \theta) = \left(2, \frac{2\pi}{3}\right) \). Then find the Cartesian coordinates of the point.

Solution

The rectangular coordinates of the point depicted below are
\[ (x, y) = (r \cos \theta, r \sin \theta) = \left(2 \cos \frac{2\pi}{3}, 2 \sin \frac{2\pi}{3}\right) = \left(-1, \sqrt{3}\right). \]

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The Cartesian coordinates of a point are \( (-1, -\sqrt{3}) \). What are its polar coordinates \( (r, \theta) \), where \( r > 0 \) and \( 0 \leq \theta < 2\pi \)?

Solution

Now \( r = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2 \) and \( \tan \theta = y/x = \sqrt{3} \). Since the point is in the third quadrant, we have \( \theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \).

Hence \( (r, \theta) = \left(2, \frac{4\pi}{3}\right) \).
Sketch the region in the plane consisting of the points whose polar coordinates satisfy $0 \leq \theta \leq \frac{\pi}{3}$.

**Solution**

Recall that $r$ can be negative as well as zero or positive. Therefore, the region is like two infinite slices of pie emanating from the pole (origin).

Sketch the region $1 \leq r < 3$, $\frac{-\pi}{4} \leq \theta \leq \frac{\pi}{4}$, in the plane.

**Solution**

This is a portion of a circular washer symmetric with the $x$-axis. Note that the $r = 3$ boundary is not included.

Find a Cartesian equation for the curve represented by the polar equation $r^2 = \theta$.

**Solution**

Take the tangent of each side of the polar equation, then substitute.

$$
\tan \left( r^2 \right) = \tan \theta \\
\tan \left( x^2 + y^2 \right) = \frac{y}{x}
$$

Find a polar equation of the curve described by the Cartesian equation $x^2 - y^2 = 1$.

**Solution**

We have $(r \cos \theta)^2 - (r \sin \theta)^2 = 1$ or $r^2 = \frac{1}{\cos^2 \theta - \sin^2 \theta}$.

Sketch the polar curve $r = 5$.

**Solution**

This is the circle of radius 5 centered at the pole (origin).

Sketch the polar curve $r = 3 \cos \theta$.

**Solution**

This is a line through the pole (origin).

MATLAB Examples

Sketch the polar curve $r = 1 - 2 \cos \theta$. 

Solution

Here is a polar plot. Use **autopolr** on your TI-89 to replicate it! (See the TAMU CALC Command Reference for details.)

```matlab
% % Stewart 820/49
% t = linspace(0, 2*pi);
r = 1 - 2*cos(t);
polar(t, r, 'c');
%
```

**s820x55**

Sketch the polar curve $r^2 = 4 \cos 2\theta$.

**Solution**

We’ll plot $r = \pm 2\sqrt{\cos 2\theta}$ for $|\theta| \leq \frac{\pi}{4}$.

```matlab
% % Stewart 820/55
% t = linspace(-pi/4, pi/4);
r1 = 2*sqrt(cos(2*t));
r2 = -r1;
polar(t, r1); hold on
polar(t, r2)
%
```

**Example A**

Find the slope of the tangent line to the cardioid $r = 1 + \cos \theta$ at the point corresponding to $\theta = \frac{\pi}{6}$.

```matlab
% % Stewart 820/55: Look, Ma, it's the infinity symbol!
%
```

**Example B**

Find the arc length of the polar curve $r = \theta^2$, $0 \leq \theta \leq 2\pi$.

**Solution**

The length of this spiral is $\frac{8}{3} \left( (\pi^2 + 1)^{3/2} - 1 \right) \approx 92.90$ cm.
\[ L = \int_{0}^{2\pi} t \cdot (t^2+4)^{1/2} \, dt \]

\[
\text{pretty}(L); \text{eval}(L)
\]

\[ \frac{2}{3} \left( \frac{8}{3} \pi + 1 \right) - \frac{8}{3} \]

\[
\text{ans} = 92.8962 \\
\%
\]

\[
t = \text{linspace}(0, 2\pi); \\
r = t^2; \\
\text{polar}(t, r) \\
\%
\]

\[
\text{echo off; diary off}
\]

Stewart 13.4 / Example B