Section 2.1

1. [40/2] For the differential equation \( y' - 2y = t^2 e^{2t} \):
   
   (a) Draw a direction field.
   
   (b) Solve the differential equation.
   
   (c) Explain how solutions behave as \( t \to \infty \).

2. [40/19] Find the solution of the initial value problem \( t^3 y' + 4t^2 y = e^{-t} \), \( y(-1) = 0 \), \( t < 0 \).

3. [40/22] Consider the initial value problem \( 2y' - y = e^{t/3} \), \( y(0) = a \).
   
   (a) Draw a direction field for the differential equation.
   
   (b) Solve the initial value problem.
   
   (c) Explain how solutions behave as \( t \to \infty \) depending on the value of \( a \).

Section 2.2

In the videos for this section, I graphed solutions explicitly. It was cumbersome! In the calculator transcripts, you’ll see graphs of implicit solutions that were done with MATLAB. Texas Instruments may implement implicit graphing this spring, perhaps announcing it at their T3 International Conference in Orlando, Florida, at the end of February. I’ll keep you posted!

1. [48/2] Solve the differential equation \( \frac{dy}{dx} = \frac{x^2}{y(1+x^3)} \).

2. [48/19] Consider the initial value problem \( \sin 2x dx + \cos 3y dy = 0 \), \( y(\pi/2) = \pi/3 \).
   
   (a) Obtain an implicit solution.
   
   (b) Implicitly plot it using MATLAB.
   
   (c) Find the interval in which the solution is defined.

3. [48/20] For \( y^2 \sqrt{1-x^2} dy = \sin^{-1}x dx \), \( y(0) = 1 \), proceed as in #2.

4. [49/22] Same as #2 for \( \frac{dy}{dx} = \frac{3x^2}{3y^3 - 4} \), \( y(1) = 0 \).

Section 2.3

1. [60/4] A tank initially holds 200 gal of water with 100 lb of salt in solution. The capacity of the tank is 500 gal. Water containing 1 lb/gal of salt enters the tank at 3 gal/min. The mixture in the tank flows out at 2 gal/min.
   
   (a) Find the amount of salt in the tank at any time \( t \) prior to the tank overflowing.
   
   (b) Find the concentration (in lb/gal) of salt in the tank when it starts to overflow.
   
   (c) Compare the concentration in (b) with the theoretical limiting concentration if the tank had infinite capacity.

2. [61/8] A person with no capital invests \( k \) dollars per year at an annual rate of return \( r \). Assume investments and compounding occur continuously; i.e., the rate of change of capital is proportional to its amount.
   
   (a) Determine the sum \( S(t) \) at any time \( t \).
   
   (b) If \( r = 7.5\% \), determine \( k \) so that $1 million will be available for retirement in 40 years.
   
   (c) If \( k = $2000/\text{year} \), determine the rate of return in order to have $1 million available in 40 years.

3. [62/13] Carbon-14 decays exponentially; i.e., its rate of change is proportional to the amount present at a given time. Given that the half-life of carbon-14 is 5730 years, answer the following.
   
   (a) Determine the rate constant; i.e., the proportionality constant.
   
   (b) Find an expression for the amount of carbon-14 at any time given its initial amount.
   
   (c) Suppose a sample of carbon-14 is determined to have 20% of its original amount. Determine the age of the sample.

4. [62/14] A population’s growth rate varies with time as follows: \( \frac{dy}{dt} = \frac{1}{5} (\frac{1}{2} + \sin t) y \).
   
   (a) If \( y(0) = 1 \) (kilopeople; i.e., 1000 people), find the time \( \tau \) at which the population has doubled. Choose other initial conditions to determine whether the doubling time \( \tau \) depends on the initial population.
   
   (b) Suppose that the growth rate is replaced by its average value \( \bar{r} \). Determine the doubling time in this case. [Turn the page for parts (c) and (d).]
(c) Suppose the term \( \sin t \) in the original differential equation is replaced by \( \sin 2\pi t \); i.e., variation in growth rate has a much higher frequency. What effect does this have on the doubling time?

(d) Plot solutions obtained in (a), (b), and (c) on a single set of axes.

5. [63/16] The temperature of an object changes at a rate proportional to the difference between the object’s temperature and that of its surroundings (the ambient temperature). [This is Newton’s law of heating/cooling.] If a cup of coffee has a temperature of 200°F when freshly poured, and 1 minute later has cooled to 190°F in a room at 70°F, determine when the temperature of the coffee is 150°F.

6. [64/20] A ball with mass 0.15 kg is thrown upward with initial velocity 20 m/s from the roof of a building 30 m high. Neglect air resistance. [Turn the page!]

(a) Find the maximum height above the ground that the ball reaches.

(b) Find the time at which the ball hits the ground.

(c) Plot graphs of velocity and position versus time.

Section 2.4

EUT: existence and uniqueness theorem

1. [76/4] Determine via the EUT for linear equations the largest interval in which the solution of the IVP is certain to exist: \( 4 - t^2 \)

2. [76/12] State where in the \( ty \)-plane the hypotheses of the EUT for nonlinear equations are satisfied for the differential equation \( \frac{dy}{dt} = \frac{(\cot t)y}{1+y} \). Draw a plot.

3. [76/14] Solve the IVP \( y' = 2ty^2 \), \( y(0) = y_0 \). Determine how the solution interval depends on \( y_0 \).

4. [76/20] Draw a direction field for \( y' = t - 1 - y^2 \) along with several solution curves. Describe how solutions behave as \( t \) increases and how their behavior depends on the initial value \( y_0 \) at \( t = 0 \).

5. [78/28] Solve \( t^2y' + 2ty - y^3 = 0 \), \( t > 0 \), a Bernoulli equation. Draw some solution curves.

Section 2.5

1. [89/12] Draw a direction field for \( y' = y^2(4 - y^2) \) along with several solution curves and equilibrium solutions. Classify the latter.

2. [90/17] (a) Consider the Gompertz equation \( y' = ry \ln(K/y) \).

(a) Solve it with initial condition \( y(0) = y_0 \).

(b) With \( r = 0.71, K = 80.5 \times 10^6, y_0/K = 0.25 \), find \( y(2) \).

(c) Using data from (b), find the time \( \tau \) at which \( y(\tau) = 0.75K \).

3. [94/28a] Let \( x' = \alpha (p-x)(q-x) \), where \( \alpha, p, q \) are positive constants with \( p \neq q \) and \( x(0) = 0 \). [This represents a chemical reaction; see textbook.]

- Determine the limiting value of \( x(t) \) as \( t \to \infty \) without solving the DE.
- Then solve the DE and verify your conjecture.

4. [94/28b: chemical reactions] Same drill as #4 for the initial value problem \( x' = \alpha (p-x)^2 \), \( x(0) = 0 \).

Section 2.6

1. [101/14] First use \texttt{deSolve} to solve the IVP \( (9x^2 + y - 1) \, dx + (x - 4y) \, dy = 0 \), \( y(1) = 0 \). Then solve it semiautomatically as follows.

(a) Note the DE has the form \( P \, dx + Q \, dy = 0 \). Show that it is exact.

(b) Then construct a potential function \( f \) for \( w = [P, Q] \) via the TAMUDFEQ 2.2 \texttt{pot} command. (It’s in the u. menu once you’ve loaded and renamed a package template.)

(c) A solution is given by \( f = C \), where \( C \) is a constant.

(d) Resolve the constant using the initial condition.

(e) Determine the domain of the solution, then graph the solution curve.

2. [102/28] Use \texttt{deSolve} to solve the differential equation \( y \, dx + (2xy - e^{-2y}) \, dy = 0 \). Next solve it semiautomatically by finding an integrating factor, then proceeding as in #1. (An implicit plot via MATLAB is included in the transcript.)