4. A general solution (verified by `dsolve`) is

\[ y = y_p + y_h = -\frac{9}{4}e^{2t} + c_1e^{3t} + c_2e^{-6t} \]

Example B

Find a general solution of \( y'' + 7y' + 10y = -4\sin3t \).

Solution

1. The homogeneous equation \( y'' + 7y' + 10y = 0 \) has characteristic equation \( 0 = r^2 + 7r + 10 = (r + 2)(r + 5) \), with roots \( r = -2, -5 \). Hence \( y_h = c_1e^{-2t} + c_2e^{-5t} \). The forcing function is \( f(t) = -4\sin3t \), which is not a solution of the homogeneous equation.

2. Let \( y_p = a\cos3t + b\sin3t \). Then

\[
\begin{align*}
y_p' &= -3a\sin3t + 3b\cos3t \\
y_p'' &= -9a\cos3t - 9b\sin3t
\end{align*}
\]

Thus \((-9a\cos3t - 9b\sin3t) + 7(-3a\sin3t + 3b\cos3t) + 10(a\cos3t + b\sin3t) = -4\sin3t \). Accordingly, we have

\[
(a + 21b)\cos3t + (4 + 21a)\sin3t = 0 = 0\cos3t + 0\sin3t
\]

3. Equating coefficients of like entities gives \( a + 21b = 0 \) and \( 4 - 21a + b = 0 \), from which \( a = \frac{2}{27}, b = -\frac{2}{27} \). For example, form the matrix system

\[
\begin{bmatrix}
1 & 21 \\
-21 & 1
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix} =
\begin{bmatrix}
0 \\
-4
\end{bmatrix}
\]

or \( \mathbf{Mc} = \mathbf{k} \). Then the MATLAB command \( \mathbf{c} = \mathbf{M}^{-1}\mathbf{k} \) solves the system. In the MATLAB Examples, I just used `solve`. Our particular solution is \( y_p = \frac{42}{27}\cos3t - \frac{2}{27}\sin3t \).

4. A general solution (verified by `dsolve`) is

\[ y = y_p + y_h = \frac{42}{27}\cos3t - \frac{2}{27}\sin3t + c_1e^{-2t} + c_2e^{-5t} \]

MATLAB Examples

First we revisit the two problems we did by hand. Then we’ll examine a variety of problems that point out various subtleties.

I purposely left comments out of the code and solutions. Try to understand what is going on from the steps we did by hand in the first two examples. In time, you ought to be able to explain to yourself (or a classmate) what is going on in your own words.

If you have questions, however, please ask! (This is especially true when you’re just starting out).
Example A, revisited [exponential forcing term]

Find a general solution of $y'' + 3y' - 18y = 18e^{2t}$.

Solution

```matlab
% % NSS4-4.5/Example A
% syms a c1 c2 r t
y = sym('y(t)');
p = poly2sym([1 3 -18], r); pretty(p)
2
r + 3 r - 18
r = solve(p)
r =

[ 3]
[ -6]
yh = c1*exp(3*t) + c2*exp(-6*t);
% L = diff(y,t,2) + 3*diff(y,t) - 18*y; pretty(L)
2
| d |
|--- y(t)| + 3 |--- y(t)| - 18 y(t)
| 2 | \dt /
yp = a*exp(2*t);
% eq0 = subs(L - 18*exp(2*t), y, yp);
% % Left - Right = 0
% eq0 = simple(eq0 / exp(2*t))
% eq0 =
-8*a-18
a = solve(eq0)
a =
-9/4
% yp = subs(yp); pretty(yp)
- 9/4 exp(2 t)
check = subs(L, y, yp)
check =
18*exp(2*t)
% y = yp + yh; pretty(y)
- 9/4 exp(2 t) + c1 exp(3 t) + c2 exp(-6 t)
sol = dsolve('D2y + 3*Dy - 18*y = 18*exp(2*t)', 't');
pretty(sol)
exp(-6 t) C2 + exp(3 t) C1 - 9/4 exp(2 t)
% echo off; diary off
```

Example B, revisited [trigonometric forcing term]

Find a general solution of $y'' + 7y' + 10y = -4 \sin 3t$.

Solution

```matlab
% % NSS4-4.5/Example B
% syms a b c1 c2 r t
y = sym('y(t)');
p = poly2sym([1 7 10], r); pretty(p)
2
r + 7 r + 10
r = solve(p)
r =

[ -2]
[ -5]
yh = c1*exp(-2*t) + c2*exp(-5*t);
% L = diff(y,t,2) + 7*diff(y,t) + 10*y; pretty(L)
2
| d |
|--- y(t)| + 7 |--- y(t)| + 10 y(t)
| 2 | \dt /
yp = a*cos(3*t) + b*sin(3*t);
% eq0 = subs(L - (-4*sin(3*t)), y, yp);
eq0 = collect(eq0, cos(3*t));
eq0 = collect(eq0, sin(3*t))
eq0 =
(4+b-21*a)*sin(3*t)+(21*b+a)*cos(3*t)
[a b] = solve(4+b-21*a, 21*b+a) % Cut & paste equations!
a =
42/221
b =
-2/221
% yp = subs(yp); pretty(yp)
42
--- cos(3 t) - 2/221 sin(3 t)
221
check = subs(L, y, yp)
check =
-4*sin(3*t)
% y = yp + yh; pretty(y)
42
--- cos(3 t) - 2/221 sin(3 t) + c1 exp(-2 t) + c2 exp(-5 t)
221
sol = dsolve('D2y + 7*Dy + 10*y = -4*sin(3*t)', 't');
pretty(sol)
exp(-5 t) C2 + exp(-2 t) C1 + --- cos(3 t) - 2/221 sin(3 t)
221
% echo off; diary off
```
Example C [polynomial forcing term]

Find a general solution of $y'' + 5y' + 6y = 4 - t^2$.

Solution

```matlab
syms a b c r t
y = sym('y(t)');
p = poly2sym([1 5 6], r); pretty(p)
2
r + 5 r + 6
r = solve(p)
r =
[-2]
[-3]
yh = c1*exp(-2*t) + c2*exp(-3*t);
L = diff(y, t, 2) + 5*diff(y, t) + 6*y; pretty(L)
/2 \
| - y(t) | + 5 | --- y(t) | + 6 y(t)
| 2 | \dt |

yp = a*cos(2*t) + b*sin(2*t);
eq0 = subs(L - 2*cos(2*t), y, yp);
eq0 = collect(eq0, cos(2*t));
eq0 = collect(eq0, sin(2*t))
eq0 = (-2*b-4*a)*sin(2*t)+(-2*a+4*b-2)*cos(2*t)
[a b c] = solve(-2*b-4*a, -2*a+4*b-2)
a =
-1/5
b = 2/5
c = 53/108
yp = subs(yp); pretty(yp)
2
- 1/6 t + 5/18 t + --- + 53
108
check = subs(L, y, yp)
check =
4-t^2
y = yp + yh;
prett(y)
2
- 1/6 t + 5/18 t + --- + c1 exp(-2 t) + c2 exp(-3 t)
108
sol = dsolve('D2y + 5*Dy + 6*y = 4 - t^2', 't');
pretty(sol)
2
C2 exp(-2 t) + exp(-3 t) C1 - 1/6 t + 5/18 t + ---
108
```

Example D [an initial value problem]

Solve the IVP $y'' + 2y' + 2y = 2 \cos 2t, \quad y(0) = -2, \quad y'(0) = 0$.

Solution

```matlab
syms a b r t
y = sym('y(t)');
p = poly2sym([1 2 2], r); pretty(p)
2
r + 2 r + 2
r = solve(p)
r =
[-1+i]
[-1-i]
yf = [exp(-t)*cos(t), exp(-t)*sin(t)];
L = diff(y, t, 2) + 2*diff(y, t) + 2*y; pretty(L)
/2 \
| - y(t) | + 2 | --- y(t) | + 2 y(t)
| 2 | \dt |

yp = a*cos(2*t) + b*sin(2*t);
eq0 = subs(L - 2*cos(2*t), y, yp);
eq0 = collect(eq0, cos(2*t));
eq0 = collect(eq0, sin(2*t))
eq0 = (-2*b-4*a)*sin(2*t)+(-2*a+4*b-2)*cos(2*t)
a =
-1/5
b = 2/5
yp = subs(yp); pretty(yp)
2/5 sin(2 t) - 1/5 cos(2 t)
check = subs(L, y, yp)
check =
2*cos(2*t)

v = [yf yp];
M = wron(v, t); % Push wron beyond its design specs!
a = M(1:2, 3) % "Can you say subvectors and

```
c = M\(b-a\)

c =

[ -9/5]
[ -13/5]

y = yp + yf*c; pretty(y)

- 13/5 \exp(-t) \sin(t) - 9/5 \exp(-t) \cos(t)
- 1/5 \cos(2t) + 2/5 \sin(2t)

\%

sol = dsolve('D2y + 2*Dy + 2*y = 2*cos(2*t)', ...
'y(0)=-2', 'Dy(0)=0', 't');
pretty(sol)

- 13/5 \exp(-t) \sin(t) - 9/5 \exp(-t) \cos(t)
- 1/5 \cos(2t) + 2/5 \sin(2t)

\%

echo off; diary off

Example E [The forcing function is a solution of the homogeneous equation.]

Find a general solution of \(y'' + 4y' + 4y = 2e^{-2t}\). Here the forcing term is also a solution of the associated homogeneous equation. Note the corresponding adjustment to the form of the particular solution!

Solution

\%

\%

\%

syms a b c c1 c2 r t

y = sym('y(t)');

p = poly2sym([1 4 4], r); pretty(p)

2
r + 4 r + 4
r = solve(p)

r =

[ -2]
[ -2]

yh = c1*exp(-2*t) + c2*t*exp(-2*t);

L = diff(y,t,2) + 4*diff(y,t) + 4*y; pretty(L)

/ 2 \  
\| --- y(t) + 4 | --- y(t) | + 4 y(t) \ /
\| 2 | \ \ \ \ \ / dt /
\y = t^2 * a*exp(-2*t)

yp =

t^2*a*exp(-2*t)

eq0 = subs(L - 2*exp(-2*t), y, yp);

eg0 = collect(eq0, exp(-2*t)) / exp(-2*t)

eq0 =
2*a-2
a = solve(2*a-2)

a =
1

\%

yp = subs(yp); pretty(yp)

2
\ t \ exp(-2 t)
\check = subs(L, y, yp)
check =

2*exp(-2*t)

y = yp + yh; pretty(y)

2
\ t \ exp(-2 t) + c1 \ exp(-2 t) + c2 \ t \ exp(-2 t)

sol = dsolve('D2y + 4*Dy + 4*y = 2*exp(-2*t)', 't');
pretty(sol)

2
\ exp(-2 t) \ C2 + t \ exp(-2 t) \ C1 + t \ exp(-2 t)

\%

echo off; diary off

Example F [multiterm forcing function]

Find a general solution of \(y'' + 16y = e^{-4t} + 3 \sin 4t\).

Your authors would have you split the forcing term up, handle each subproblem separately, then superimpose the solutions. That’s fine, perhaps even preferable when doing things by hand. With massive firepower on tap, however, there is no need for this.

Note once again that a part of the forcing term is also a solution of the associated homogeneous equation. Accordingly, we make an adjustment in the form of that part of the particular solution.

Finally, with a little reassessment, we see that \texttt{dsolve}'s solution is equivalent to ours.

Solution

\%

\%

\%

syms a b c c1 c2 r t C2 K

y = sym('y(t)');

p = poly2sym([1 0 16], r); pretty(p)

2
r + 4 r + 4
r = solve(p)

r =

[ -4*i]
[ -4*i]

yh = c1*cos(4*t) + c2*sin(4*t);

L = diff(y,t,2) + 16*y; pretty(L)

/ 2 \  
\| --- y(t) + 16 | y(t) \ /
\| 2 | \ \ \ / dt /
\y = t^2 * a*cos(4*t) + t*b*exp(-4*t) + c*sin(4*t))

eq0 = subs(L - (exp(-4*t) + 3*sin(4*t)), y, yp);

eg0 = collect(eq0, cos(4*t))

eq0 = collect(eq0, sin(4*t))

eq0 = collect(eq0, cos(4*t))
Example G

Find a general solution of $y'' + 5y' + 4y = te^{-t}$. There are subtleties here too, but by now you're a pro...