Spring 2003  Math 308/501–502
6 Theory of Higher-Order Linear ODEs
6.3 Method of Undetermined Coeffs
Wed, 01/Oct ©2003, Art Belmonte

Summary
A nonhomogeneous linear ODE of order \( n \) with real constant coefficients has the form \( L[y] = \sum_{k=0}^{n} a_k y^{(k)} = f \). Here \( t \) (or \( x \)) is the independent variable and the nonhomogeneity \( f \neq 0 \) is known as the “forcing function;” it may consist of one or several terms.

The associated homogeneous ODE is \( L[y] = \sum_{k=0}^{n} a_k y^{(k)} = 0 \). A general solution \( y_h \) of this is obtained via §6.2 methods.

General solution of the nonhomogeneous equation
If \( y_p \) is a particular solution of the nonhomogeneous equation and \( y_h \) is a general solution to the associated homogeneous equation, then a general solution of the nonhomogeneous equation is given by \( y = y_p + y_h \).

Superposition Principle
For \( k = 1, 2, \ldots, M \), let \( y_{p_k} \) be a solution of \( L[y] = f_k \). Then for any constants \( c_1, \ldots, c_M \), the function \( y_p = \sum_{k=1}^{M} c_k y_{p_k} \) solves the differential equation \( L[y] = \sum_{k=1}^{M} c_k f_k \). (This follows immediately from the fact that \( L \) is a linear differential operator.)

Method of Undetermined Coefficients
If the forcing function \( f(t) = \sum_{k=1}^{M} f_k(t) \) is a sum of products of real polynomials, sines, cosines, and/or exponentials, then the following method produces a general solution of \( L[y] = f(t) \). Each \( f_k(t) \) must have the form

\[ e^{\alpha t} (p_u(t) \cos \beta t + q_v(t) \sin \beta t) \]

where \( p_u \) and \( q_v \) are polynomials of degrees \( u \) and \( v \), whereas \( \alpha \) and \( \beta \) are real constants. Now \( \alpha \) or \( \beta \) may be 0, \( p_u \) or \( q_v \) may be constants, and these may all vary with the index \( k \). Often, \( M \) is 1; i.e., there is a single term in the forcing function.

1. First obtain a general solution \( y_h \) of \( L[y] = 0 \).
2. For each \( k \), determine a particular solution \( y_{p_k} \) of \( L[y] = f_k(t) \), as follows. (If \( M = 1 \), just use \( y_p \) for \( y_{p_1} \).)
   (a) Form \( y_{p_k} = \alpha t e^{\beta t} (P_N(t) \cos \beta t + Q_N(t) \sin \beta t) \), where \( N = \max(u, v) \) and \( s \) is the smallest nonnegative integer so that no term of the particular solution \( y_{p_k} \) is a solution of the corresponding homogeneous equation \( L[y] = 0 \). Here \( P_N \) and \( Q_N \) are general polynomials of degree \( N \) which have undetermined coefficients; i.e., the coefficients are symbolic at this stage. From \( y_{p_k} \), compute derivatives \( y_{p_k}^{(j)} \), \( j = 1, 2, \ldots, n \), then substitute into \( L[y] = f_k(t) \).
   (b) Collect and equate like terms. This results in a linear system of equations. Solve for the undetermined coefficients. Now you know \( y_{p_k} \).
3. Let \( y_p = \sum_{k=1}^{M} y_{p_k} \). The general solution of \( L[y] = f(t) \), via superposition, is \( y = y_p + y_h \).

By hand or with MATLAB, you’re a winner!
The hand work involved in computing derivatives, substitution, collecting terms, and solving linear systems can be tedious and error-prone. It lends itself quite well, however, to machine power, as you’ll observe in the MATLAB Examples.

You may have felt that the formulation for \( y_p \) given above is overly complicated. Take solace in the fact that under that stated formulation there is only one case (the general case). What’s more, there is no guesswork as to the correct form of \( y_p \). With practice, you’ll get it right every time!

Note that in all examples (hand or MATLAB), we carry the work through to a full general solution (or to the unique solution of an initial value problem). Please do this in your homework problems as well.

Hand Example
In general, when dealing with ODEs of order 3 or higher, we use MATLAB! Hand computations are too lengthy. Here’s one done by hand for grins. You’ll see what I mean. (Please see the Section 4.5 lecture handout for two more hand examples.)

Find a general solution of \( y''' - 2y'' - 5y' + 6y = e^t + x^2 \).
Solution

Even with a TI-89 to help, this is cumbersome!

1. The homogeneous equation $y''' - 2y'' - 5y' + 6y = 0$ has characteristic equation $r^3 - 2r^2 - 5r + 6 = 0$, with roots $r = -2, 1, 3$. Hence $y_h = c_1 e^{-2x} + c_2 e^x + c_3 e^{3x}$.

2. Let $y_p = x \cdot a e^x + bx^2 + cx + d$. Its derivatives are $y'_p = a(x + 1)e^x + 2bx + c$, $y''_p = a(x + 2)e^x + 2b$, and $y'''_p = a(x + 3)e^x$. Substitute into the differential equation.

\[
a(x + 3)e^x - 2(a(x + 2)e^x + 2b) - 5(a(x + 1)e^x + 2bx + c) + 6(axe^x + bx^2 + cx + d) = e^x + x^2.
\]

After collecting, we have $-6ae^x + 6bx^2 + (6c - 10b)x - 4b - 5c + 6d = e^x + x^2$.

3. Equating coefficients yields a system of four equations:

\[
-6a = 1, \quad 6b = 1, \quad 6c - 10b = 0, \quad -4b - 5c + 6d = 0.
\]

Solving these gives $a = -\frac{1}{6}$, $b = \frac{5}{18}$, $c = \frac{5}{18}$, $d = \frac{37}{108}$. Thus $y_p = -\frac{1}{6} x e^x + \frac{1}{6} x^2 + \frac{5}{18} x + \frac{37}{108} e^x$ is our particular solution.

4. A general solution (verified by `dsolve`) is $y = y_p + y_h$.

\[
y_p = -\frac{1}{6} x e^x + \frac{1}{6} x^2 + \frac{5}{18} x + \frac{37}{108} e^x + c_1 e^{-2x} + c_2 e^x + c_3 e^{3x}.
\]

MATLAB Examples

Bring the requisite machine power to bear upon the problem!

337/1 [revisited]

Find a general solution of $y''' - 2y'' - 5y' + 6y = e^x + x^2$.

Solution

```matlab
% NSS4-337/1
% sym a b c d c1 c2 c3 r x
y = sym('y(x)');
p = poly2sym([1 -2 -5 6], r); pretty(p)
3 2
r = solve(p)
r =
[ -1]
[ -2]
[ 3]
yh = c1*exp(x) + c2*exp(-2*x) + c3*exp(3*x); %
L = diff(y,x,3) - 2*diff(y,x,2) - 5*diff(y,x) + 6*y;
pretty(L)
```

337/31

Solve the IVP $y''' + 2y'' - 9y' - 18y = -18x^2 - 18x + 22$;
$y(0) = -2, \quad y'(0) = -8, \quad y''(0) = -12$. (Please turn the page for the solution.)
Solution

```matlab
% NSS4-337/31
% syms a b c c1 c2 c3 r x
y = sym('y(x)');
p = poly2sym([1 2 -9 -18], r); pretty(p)
   3  2
r + 2 r - 9 r - 18
r = solve(p)
   [-2]
   [-3]
   [ 3]
yf = [exp(-2*x), exp(-3*x), exp(3*x)];
L = diff(y,x,3) + 2*diff(y,x,2) - 9*diff(y,x) - 18*y
pretty(L)
   / 3 \  / 2 \  /d \|d \|d \|d \|
   |--- y(x)| + 2 |--- y(x)| - 9 |--- y(x)| - 18 y(x)
   \dx / \dx / \dx /
yp = a*x^2 + b*x + c;
eq0 = subs(L, y, yp);
eq0 = collect(eq0, x)
eq0 =
(18-18*a)*x^2+(18-18*b-18*a)*x-22+4*a-18*c-9*b
[a b c] = solve(18-18*a, 18-18*b-18*a, ...
   -22+4*a-18*c-9*b)
a =
   1
b =
   0
c =
   -1
yp = subs(yp); pretty(yp)
   2
   x - 1
% We're done with a,b,c (for now)...
check = subs(L, y, yp)
   -18*x^2-18*x+22
v = [yf yp];
```

337/29

Find a general solution of $z''' - 2z'' + z' = x - e^x$.

Solution

```matlab
% NSS4-337/29
% syms a b c c1 c2 c3 r x k1 k2 k3 c1 c2 c3
z = sym('z(x)');
p = poly2sym([1 2 -9 31], r); pretty(p)
   3  2
r + 2 r + r
r = solve(p)
   [-2]
   [-3]
   [ 3]
M = wron(v, x); M = subs(M, x, 0) % PUSH WRON!
M =
   1  1  1
   [-2] 3  0
   4  9  2
   [-8] 27 27 0
% Redefining a, b, c to resolve ICs.
a = M(1:3, 4)
a =
   [-1]
   [ 0]
   [ 2]
M = M(1:3, 1:3)
M =
   1  1  1
   [-2] 3  3
   4  9  9
b = sym([-2; -8; -12])
b =
   [-2]
   [-8]
   [-12]
c = M\(b-a)
c =
   [ 1]
   [ 0]
   [-2]
y = yp + yf*c; pretty(y)
   2
   x - 1 + exp(-2 x) - 2 exp(3 x)
% Beauty, eh?
% sol = dsolve(...
%   'D3y + 2*D2y - 9*Dy - 18*y - 18*x^2 -18*x + 22', ...
%   'y(0)=-2', 'Dy(0)=-8', 'D2y(0)=-12', 'x');
pretty(sol)
   2
   x - 1 + exp(-2 x) - 2 exp(3 x)
% echo off; diary off
```
\begin{verbatim}
\textbf{zh} = \texttt{c1 + c2*exp(x) + c3*x*exp(x);}
\%
\textbf{L} = \texttt{diff(z,x,3) - 2*diff(z,x,2) + diff(z,x);}  
\texttt{pretty(L)}
\begin{verbatim}
L = \frac{d^3 z(x)}{dx^3} - 2 \frac{d^2 z(x)}{dx^2} + \frac{dz(x)}{dx}
\end{verbatim}
\texttt{zp} = \texttt{x * \{a*x + b\} + x^2 * c*exp(x);} 
\texttt{eq0} = \texttt{subs(L - (x - exp(x)), z, zp);} 
\texttt{eq0} = \texttt{collect(eq0, exp(x));}
\texttt{eq0} = \texttt{collect(eq0, x)}
\texttt{eq0 = } 
\texttt{(-1+2*a)*x+(2*c+1)*exp(x)-4*a+b}
\texttt{[a b c] = solve(-1+2*a, 2*c+1, -4*a+b)}
\texttt{a = 1/2}
\texttt{b = 2}
\texttt{c = -1/2}

\texttt{zp = subs(zp); pretty(zp)}
\begin{verbatim}
2
x \left(\frac{1}{2} \cdot x + 2\right) - \frac{1}{2} \cdot x \cdot \texttt{exp(x)}
\end{verbatim}
\texttt{check = subs(L, z, zp)}  
\texttt{check = x-exp(x)}
\texttt{z = zp + zh; z = collect(z, exp(x));}
\texttt{pretty(z)}
\begin{verbatim}
2
\left(-1/2 \cdot x + c2 + c3 \cdot x\right) \cdot \texttt{exp(x)} + x \left(\frac{1}{2} \cdot x + 2\right) + c1
\end{verbatim}
\texttt{sol = dsolve('D3y - 2*D2y + Dy = x- exp(x)', 'x')} 
\texttt{sol = collect(sol, exp(x)); pretty(sol)}
\begin{verbatim}
2
\left(C2 + C1(x - 1) + x - 1 - 1/2 \cdot x\right) \cdot \texttt{exp(x)} + 1/2 \cdot x + C3 + 2x
\end{verbatim}
\%
\texttt{sol = subs(sol, [C1 C3], [k3-1, k1]);}
\texttt{sol = simplify(sol); sol = collect(sol, exp(x));}
\texttt{pretty(sol)}
\begin{verbatim}
2
\left(C2 + k3 \cdot x - k3 - 1/2 \cdot x\right) \cdot \texttt{exp(x)} + 1/2 \cdot x + k1 + 2 x
\end{verbatim}
\%
\texttt{sol = subs(sol, C2, k2+k3); pretty(sol)}
\begin{verbatim}
2
(k2 + k3 \cdot x - 1/2 \cdot x) \cdot \texttt{exp(x)} + 1/2 \cdot x + k1 + 2 x
\end{verbatim}
\%
\texttt{Voila!}
\%
\texttt{And that retires the side. Good night!}
\%
\texttt{echo off; diary off}
\end{verbatim}