7 The Laplace Transform

**Summary**

**Definition**

The **convolution** of two piecewise continuous functions \( f \) and \( g \) is

\[
(f * g)(t) = \int_0^t f(t - v)g(v) \, dv
\]

**Properties of the convolution**

For piecewise continuous functions \( f, g, \) and \( h \), we have

1. \( f * g = g * f \) (The convolution is commutative.)
2. \( f * (g + h) = f * g + f * h \) (It is distributive.)
3. \( (f * g) * h = f * (g * h) \) (It is associative.)
4. \( f * 0 = 0 \) (A function convolved with zero is zero.)

**The Convolution Theorem**

If \( f \) and \( g \) are piecewise continuous functions on \([0, \infty)\) of exponential order with Laplace transforms \( \mathcal{L}\{f\} = F(s) \) and \( \mathcal{L}\{g\} = G(s) \) then

\[
\mathcal{L}\{f * g\} = F(s)G(s).
\]

**Initial value problems**

The **transfer function** \( H(s) \) of a linear system \( L[y] = g(t) \) with all initial conditions zero is \( H(s) = Y(s)/G(s) \); i.e., the Laplace transform of the output function \( y(t) \) divided by that of the input function \( g(t) \). If \( L[y] \) has constant coefficients, then \( H(s) \) is the reciprocal of \( p(s) \), the characteristic polynomial of the corresponding homogeneous equation; i.e., \( H(s) = 1/p(s) \).

The **impulse response function** is the inverse Laplace transform of the transfer function; i.e., \( h(t) = \mathcal{L}^{-1}\{H(s)\} \). Physically, it describes the solution when a mass-spring system is struck by a hammer or a baseball is struck by a bat. (In Section 7.8 we'll study impulses and the Dirac delta function, which model these physical situations.)

**Theorem: Solution using the impulse response function**

Let \( I \) be an open interval containing the origin, with \( a, b, \) and \( c \) constants and \( g \) continuous on \( I \). The unique solution to the IVP

\[
L[y] = ay'' + by' + cy = g, \quad y(0) = y_0, \quad y'(0) = y_1
\]

is

\[
y(t) = (h * g)(t) + y_k(t) = \int_0^t h(t - v)g(v) \, dv + y_k(t).
\]

Here \( h \) is the impulse response function and \( y_k \) is the unique solution to \( L[y] = 0, \quad y(0) = y_0, \quad y'(0) = y_1 \).

**Hand Examples**

**405/1**

Use the Convolution Theorem to obtain a formula for the solution to

\[
y'' - 2y' + y = g(t), \quad y(0) = -1, \quad y'(0) = 1.
\]

**Solution**

We employ the usual 4-step procedure, using the Convolution Theorem in Step 4.

1. Take the Laplace transform of each side of the DE.

\[
(s^2Y(s) - s y(0) - y'(0)) - 2(sY(s) - y(0)) + Y(s) = G(s)
\]

2. Substitute for the initial conditions.

\[
(s^2Y(s) + s - 1) - 2(sY(s) + 1) + Y(s) = G(s)
\]

3. Solve for \( Y(s) \), the Laplace transform of \( y(t) \).

\[
\begin{align*}
(s^2 - 2s + 1)Y(s) &= G(s) - s + 3 \\
Y(s) &= \frac{G(s) - s + 3}{(s - 1)^2} \\
&= \frac{G(s)}{(s - 1)^2} + \frac{a}{(s - 1)} + \frac{b}{(s - 1)^2}, \quad \text{via cpf}
\end{align*}
\]

4. Take the inverse Laplace transform of \( Y(s) \) to obtain \( y(t) \).

\[
y(t) = 2te^t - e^t + (te^t * g) = 2te^t - e^t + \int_0^t (t - v)e^{t-v} g(v) \, dv.
\]

**405/5**

Use the Convolution Theorem to find \( \mathcal{L}^{-1}\left\{\frac{1}{s(s^2 + 1)}\right\} \).
Solution

We have
\[ L^{-1} \left\{ \left( \frac{1}{s} \right) \left( \frac{1}{s^2 + 1} \right) \right\} = 1 * \sin(t) = \int_0^t 1 \sin(v) \, dv = 1 - \cos t. \]

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Find the Laplace transform of \( f(t) = \int_0^t \sin(t - v) \cdot e^v \, dv. \)

Solution

Apply the Convolution Theorem.
\[ \mathcal{L} \{ \sin t * e^t \} = \left( \frac{1}{s^2 + 1} \right) \left( \frac{1}{s - 1} \right) = \frac{1}{(s - 1)(s^2 + 1)} \]

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Solve the integral equation \( y(t) + \int_0^t (t - v)^2 y(v) \, dv = t^3 + 3. \)

Solution

1. Take the Laplace transform of each side of the equation.
\[ Y(s) + \frac{2}{s^2} Y(s) = \frac{6}{s^3} + \frac{3}{s} \]

2. (There are NO initial conditions. Skip this step!)

3. Solve for \( Y(s) \), the Laplace transform of \( y(t) \).
\[ \begin{align*}
Y(s) &= \frac{6}{s^3} + \frac{3}{s} \\
\frac{s^3 + 2}{s^3} Y(s) &= \frac{3s^3 + 6}{s^4} = \frac{3(s^3 + 2)}{s^4} \\
\frac{3}{s^4} Y(s) &= \frac{3}{s} \\
Y(s) &= \frac{3}{s} \\
Y(s) &= \frac{3}{s}
\end{align*} \]

4. Take the inverse Laplace transform of \( Y(s) \) to obtain \( y(t) \).
\[ y(t) = 3 \]

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Solve the integro-differential equation
\[ y'(t) - 2 \int_0^t e^{t-v} y(v) \, dv = t, \quad y(0) = 2. \]

Solution

1. Take the Laplace transform of each side of the equation.
\[ (sY(s) - y(0)) - 2 \frac{1}{s-1} Y(s) = \frac{1}{s^2} \]

2. Substitute for the initial conditions.
\[ sY(s) - 2 - 2 \frac{1}{s-1} Y(s) = \frac{1}{s^2} \]

3. Solve for \( Y(s) \), the Laplace transform of \( y(t) \).
\[ \begin{align*}
\left( \frac{s - 2}{s - 1} \right) Y(s) &= \frac{1}{s^2} + 2 \\
\frac{s^2 - s - 2}{s - 1} Y(s) &= \frac{2s^2 + 1}{s^2} \\
Y(s) &= \frac{(2s^2 + 1)(s - 1)}{s^2(s^2 - s - 2)} \\
Y(s) &= \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s + 1} + \frac{d}{s - 2} \\
Y(s) &= \frac{-3}{s} + \frac{1}{s^2} + \frac{2}{s + 1} + \frac{3}{s - 2}, \quad \text{via \( \text{cpf} \)}
\end{align*} \]

4. Take the inverse Laplace transform of \( Y(s) \) to obtain \( y(t) \).
\[ y(t) = -3 + \frac{1}{2} t + 2e^{-t} + \frac{3}{4} e^{2t} \]

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A linear system is governed by the initial value problem
\[ L[y] = y'' - y' - 6y = g(t), \quad y(0) = 1, \quad y'(0) = 8 \quad(*) \]

- Find the transfer function \( H(s) \) for the system.
- Compute the impulse response function \( h(t) \).
- Solve \( L[y] = 0, \; y(0) = 1, \; y'(0) = 8 \).
- Give a formula for the solution to the IVP (*)

Solution

- The transfer function is
\[ H(s) = \frac{1}{s^2 - s - 6} = \left( \frac{1}{s + 2} \right) \left( \frac{1}{s - 3} \right). \]

- The impulse response function is
\[ h(t) = \mathcal{L}^{-1} \{ H(s) \} = e^{-2t} \cdot e^{3t} = \int_0^t e^{-2(t-v)} e^{3v} \, dv = e^{3t} - e^{-2t}. \]
- Solve $L[y] = y'' - y' - 6y = 0$, $y(0) = 1$, $y'(0) = 8$, using (say) Chapter 4 techniques. The characteristic equation, $0 = r^2 - r - 6 = (r + 2)(r - 3)$, has roots $r = -2, 3$. A general solution of $L[y] = 0$ is $y_h = c_1 e^{-2t} + c_2 e^{3t}$. Its derivative is $y'_h = -2c_1 e^{-2t} + 3c_2 e^{3t}$. Substituting the ICs gives $c_1 + c_2 = 1$ and $-2c_1 + 3c_2 = 8$, or $\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$, whence $\mathbf{c} = [c_1; c_2] = [-1; 2]$. Hence the unique solution of $L[y] = 0$ satisfying the ICs is $y_k = 2e^{3t} - e^{-2t}$.

- Via the last theorem in the summary, the solution of (*) is

$$y(t) = (h * g)(t) + y_k(t) = 2e^{3t} - e^{-2t} + \int_0^t \frac{1}{5} \left( e^{3(t-v)} - e^{-2(t-v)} \right) g(v) \, dv$$