9.6 Complex Eigenvalues

**Summary**

Let $A$ be an $n \times n$ real matrix and let $p(r)$ be its characteristic polynomial. The Fundamental Theorem of Algebra guarantees that $p(r)$ factors as $(-1)^n \prod_{j=1}^{k} (r - r_j)^{q_j}$, where $r_1, \ldots, r_k$ are the distinct eigenvalues of $A$ and $\sum_{j=1}^{k} q_j = n$.

**Definitions**

- The **algebraic multiplicity** of $r_j$ is $q_j$; i.e., the number of times $r - r_j$ appears in the factorization of $p(r)$.

- The **geometric multiplicity** of $r_j$ is $d_j$, dimension of the eigenspace of $r_j$; i.e., the dimension of nullspace of $A - r_j I$.

We always have $1 \leq d_j \leq q_j$. If we’re lucky, we have $d_j = q_j$ for $j = 1, \ldots, k$. For in this case we have $\sum_{j=1}^{k} d_j = \sum_{j=1}^{k} q_j = n$ and thus a full set of linearly independent eigenvectors from which to construct a fundamental solution set. (If we’re not lucky, we resort to the Jordan canonical form.)

**Facts**

- If $r$ is an eigenvalue of $A$ with associated eigenvector $v$, then $x = e^{rt}v$ is a solution of $x' = Ax$. (If $r$ is real, then $x$ is real-valued. If $r$ is complex, then $x$ is complex-valued.)

- If $r_1, \ldots, r_k$, are distinct eigenvalues of $A$ with associated eigenvectors $v_1, \ldots, v_k$, then these eigenvectors are linearly independent.

- If $A$ has $n$ distinct eigenvalues $r_1, \ldots, r_n$, with associated eigenvectors $v_1, \ldots, v_n$, then $x_k(t) = e^{r_k t}v_k$, form a fundamental set of solutions for the system $x' = Ax$.

- Suppose that $r_1, r_2 = \alpha \pm \beta i$ (where $\beta > 0$) is a pair of complex conjugate eigenvalues of $A$. Let $w = a + ib$ be a complex eigenvector associated with the eigenvalue $r_1 = \alpha + \beta i$. (Here $a$ and $b$ are the real and imaginary parts of $w$, respectively.) Then a pair of real solutions of $x' = Ax$ is given by the real and imaginary parts of the complex solution $z = e^{(\alpha + \beta i)t}w$; namely,

  \[
  \begin{align*}
  x &= e^{\alpha t} \cos \beta t \ a - e^{\alpha t} \sin \beta t \ b \\
  y &= e^{\alpha t} \sin \beta t \ a + e^{\alpha t} \cos \beta t \ b.
  \end{align*}
  \]

  Moreover, these two real solutions are linearly independent.

**Hand Examples**

In today’s hand examples, we’ll focus on the broad sweep of the problem. That is, assuming we have the necessary **eigenpairs** (i.e., pairs of eigenvalues with associated eigenvectors), we simply form the solution. All computations are relegated to the corresponding problem in the MATLAB Examples, q.v.

**Example A**

Find the real and imaginary parts of $z(t) = e^{(1+i)t} \begin{bmatrix} -1 + i \\ 2 \end{bmatrix}$.

**Solution**

Recall Euler’s formula from Section 4.3: $e^{i\theta} = \cos \theta + i \sin \theta$. Therefore, $e^{(\alpha + i \beta)t} = e^{\alpha t} e^{i \beta t} = e^{\alpha t} (\cos \beta t + i \sin \beta t)$.

Using complex arithmetic, we have

\[
\begin{align*}
  e^{(1+i)t} &= e^t \left( \cos t + i \sin t \right) \begin{bmatrix} -1 + i \\ 2 \end{bmatrix} \\
  &= e^t \left( \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} -1 + i \\ 2 \end{bmatrix} \right) \\
  &= e^t \left( \begin{bmatrix} \cos t - \sin t & i \cos t + \sin t \\ -\sin t & \cos t + \sin t \end{bmatrix} \begin{bmatrix} -1 + i \\ 2 \end{bmatrix} \right) \\
  &= e^t \left( \begin{bmatrix} -\cos t - \sin t & i \cos t + \sin t \\ 2 \sin t & \cos t - \sin t \end{bmatrix} \right) \begin{bmatrix} -1 + i \\ 2 \end{bmatrix}.
\end{align*}
\]

These computations are both tedious and error prone. Accordingly, I wrote a MATLAB routine called **rise** ("Real Independent Solutions from a Complex eigenpair"). It extracts the real and imaginary parts of $e^{rt}w$, where $r$ and $w$ are a complex eigenpair. This is done in a fully automatic fashion. Feel free to use it! (See the corresponding MATLAB problem.)

**Example B**

Find a fundamental set of real solutions for $u' = Au$, where

\[
  A = \begin{bmatrix} -1 & 1 \\ -5 & -5 \end{bmatrix}.
\]
Solution

1. The eigenvalues of $A$ are $r_1, r_2 = -3 \pm i$.

2. An eigenvector associated with $r_1 = -3 + i$ is 
   \[ v = \begin{bmatrix} 1 \\ -2 + i \end{bmatrix}. \]

3. A complex solution is $z = e^{(-3+i)t} \begin{bmatrix} 1 \\ -2 + i \end{bmatrix}$. Via rise, the real and imaginary parts of $z(t)$ are
   \[ x(t) = e^{-3t} \begin{bmatrix} \cos t \\ -2 \cos t - \sin t \end{bmatrix}, \]
   \[ y(t) = e^{-3t} \begin{bmatrix} \sin t \\ -2 \sin t + \cos t \end{bmatrix}. \]

These form a fundamental set of real solutions of the system.

Example C

Find the solution of Example B that satisfies $u(0) = [1; -5]$.

Solution

A fundamental matrix for the system in Example B is

\[ U = [x, y] = e^{-3t} \begin{bmatrix} \cos t \\ -2 \cos t - \sin t \end{bmatrix} e^{-3t} \begin{bmatrix} \sin t \\ -2 \sin t + \cos t \end{bmatrix}. \]

and a general solution is $u = Uc$ where $c = [c_1; c_2]$.

Given $u_0 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, let $M = U(0) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$. Then

\[ u_0 = u(0) = U(0)c = Mc, \] from which we conclude

\[ c = M^{-1}u_0 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \]

Therefore, the unique solution of the initial value problem is

\[ u = Uc = [x, y] \begin{bmatrix} 1 \\ -3 \end{bmatrix} = x - 3y = e^{-3t} \begin{bmatrix} \cos t - 3 \sin t \\ 5 \sin t - 5 \cos t \end{bmatrix}. \]

MATLAB Examples

Example A [revisited]

Find the real and imaginary parts of $z(t) = e^{(1+i)t} \begin{bmatrix} -1 + i \\ 2 \end{bmatrix}$.

Solution

% NSS4-9.6/Example C
% Given a complex eigenvalue-eigenvector pair, % the routine RISC (written, however, in lowercase) % constructs two Real Independent Solutions

% from this Complex eigenpair (r, w). This saves % you the trouble of resorting to Euler’s formula % and doing a lot of complex arithmetic!
% The solutions returned are actually the real % and imaginary parts of the complex solution % exp(r*t)^c * w.
% [x,y] = risc(r+i, [-1+i; 2])

Example B [revisited]

Find a fundamental set of real solutions for $u' = Au$, where $A = \begin{bmatrix} -1 & 1 \\ -5 & -5 \end{bmatrix}$.

Solution

% NSS4-9.6/Example B
% syms t
A = sym([-1 1; -5 -5])
A =
\begin{bmatrix} -1, 1 \\ -5, -5 \end{bmatrix}
[V,D] = eig(A)
V =
\begin{bmatrix} -2/5-1/5*i, -2/5+1/5*i \\ 1, 1 \end{bmatrix}
D =
\begin{bmatrix} -3+i, 0 \\ 0, -3-i \end{bmatrix}
% While the eigenvectors returned by eig are fine,
% here we choose a nonzero multiple of the one
% associated with -3+i so as to match the solution
% in the back of the [Spring 2003] textbook.
% r = sym(-3+i);
w = sym([-2/5 - 1/5*i; 1] * (-2+i))
w =
\begin{bmatrix} 1 \\ (-2)+(1)*i \end{bmatrix}
% [x,y] = risc(r,w)
%
Example C [revisited]

Find the solution of Example B that satisfies \( u(0) = [1; -5] \).

Solution

\[
x = 
\begin{bmatrix}
\exp(-3t) \cdot \cos(t) \\
\exp(-3t) \cdot (-2 \cdot \cos(t) \cdot \sin(t))
\end{bmatrix}
\]
\[
y = 
\begin{bmatrix}
\exp(-3t) \cdot \sin(t) \\
\exp(-3t) \cdot (-2 \cdot \sin(t) + \cos(t))
\end{bmatrix}
\]

\% echo off; diary off

Example D

Find the unique solution to the initial value problem

\[
y' = \begin{bmatrix} 6 & -6 & -6 & -8 \\ -8 & -6 & -6 & -8 \\ -1 & 7 & -10 & -9 \\ -8 & 6 & 6 & 6 \end{bmatrix} y, \quad y(0) = \begin{bmatrix} -2 \\ -1 \\ 6 \\ -5 \end{bmatrix}
\]

Solution

\[
x = 
\begin{bmatrix}
\exp(-3t) \cdot \cos(t) \\
\exp(-3t) \cdot (-2 \cdot \cos(t) \cdot \sin(t))
\end{bmatrix}
\]
\[
y = 
\begin{bmatrix}
\exp(-3t) \cdot \sin(t) \\
\exp(-3t) \cdot (-2 \cdot \sin(t) + \cos(t))
\end{bmatrix}
\]

\% echo off; diary off
\[ \begin{bmatrix} 0 \\ \end{bmatrix} \]

\[ y_3 = risc(-1+i, [2; 2; 3-i; -2]) \]

\[ y_3 = \begin{bmatrix} 2 \exp(-t) \cos(3t) \\ 2 \exp(-t) \cos(3t) \\ \exp(-t) (3 \cos(3t) + \sin(3t)) \\ -2 \exp(-t) \cos(3t) \end{bmatrix} \]

\[ y_4 = \begin{bmatrix} 2 \exp(-t) \sin(3t) \\ 2 \exp(-t) \sin(3t) \\ \exp(-t) (3 \sin(3t) - \cos(3t)) \\ -2 \exp(-t) \sin(3t) \end{bmatrix} \]

\%

\[ Y = [y_1 \ y_2 \ y_3 \ y_4]; \]

\[ M = \text{sym}(\text{subs}(Y, t, 0)) \]

\[ M = \begin{bmatrix} 11 & 9 & 2 & 0 \\ 8 & 7 & 2 & 0 \\ 0 & 5 & 3 & -1 \\ 5 & 0 & -2 & 0 \end{bmatrix} \]

\[ b = [-2; -1; 6; -5] \]

\[ b = \begin{bmatrix} -2 \\ -1 \\ 6 \\ -5 \end{bmatrix} \]

\[ c = M \backslash b \]

\[ c = \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \end{bmatrix} \]

\text{syms} \ t \ \text{unreal} \ % \text{Ax} \ \text{tildes!} \]

\[ y = \text{simple}(Y \cdot c); \text{pretty}(y) \]

\[ y = \begin{bmatrix} -2 \exp(-2t) - 2 \exp(-t) \sin(3t) \\ -\exp(-2t) - 2 \exp(-t) \sin(3t) \\ 5 \exp(-2t) - \exp(-t) (3 \sin(3t) - \cos(3t)) \\ -5 \exp(-2t) + 2 \exp(-t) \sin(3t) \end{bmatrix} \]

\%

\text{echo off}; \text{diary off}