**Method of Undetermined Coefficients for Systems**

Let $A$ be an $n \times n$ real constant matrix and $f$ an $n \times 1$ column vector whose elements are sums of products of real polynomials, sines, cosines, and/or exponentials involving the independent variable $t$. We may use the method of undetermined coefficients to find a particular solution $x_p$ to the nonhomogeneous linear system $x' = Ax + f$ or $L[x] = x' - Ax = f$. The undetermined coefficients involved are now symbolic vector constants.

Moreover, in case an element of $f$ is replicated in a general solution of the associated homogeneous linear system $L[x] = 0$, the original choice for a particular solution must not only be multiplied by the smallest positive integer power of $t$ so that no term of the particular solution $x_p$ is a solution of the homogeneous equation $L[x] = 0$, but also by all lower nonnegative integer powers of $t$ as well. This is easier said than done. Indeed, when this level of complexity is reached, it is simpler to resort to **variation of parameters**, the other technique for finding particular solutions that we encountered. This will be discussed later in lecture handout 9.7V.

**Superposition Principle**

For $k = 1, 2, \ldots, M$, let $x_{p_k}$ be a solution of $L[x] = f_k$. Then for any constants $c_1, \ldots, c_M$, the function $x_p = \sum_{k=1}^{M} c_k x_{p_k}$ solves the nonhomogeneous linear system $L[y] = \sum_{k=1}^{M} c_k f_k$. (This follows immediately from the fact that $L$ is a linear operator.)

**Hand Examples**

In our first example, we’ll do things soup-to-nuts by hand. In the next example, we’ll assume we have the necessary eigenpairs (i.e., pairs of eigenvalues with associated eigenvectors) so as to rapidly form a general solution of the associated homogeneous system. Then we’ll proceed to the main course: finding a particular solution of the nonhomogeneous system.

---

**Solution**

Here is our overall solution strategy.

1. Find a general solution $x_h$ to the associated homogeneous system $x' = Ax$.

2. Find a particular solution $x_p$ to the nonhomogeneous system $x' = Ax + f$.

3. Form a general solution of the nonhomogeneous system: $x = x_p + x_h$.

Along the way, we’ll flesh out details. Let’s get the party started.

1. **Computation of $x_h$.**

   - **Eigenvalues of $A$.** Solve $\det(A - rI) = 0$.
     \[
     \begin{vmatrix}
     1 - r & 1 \\
     4 & 1 - r
     \end{vmatrix}
     = r^2 - 2r - 3 = (r + 1)(r - 3) = 0,
     \]
     whence $r = -1, 3$.

   - **Associated Eigenvectors.** Find a nonzero vector in the nullspace of the RREF of $A - rI$.
     - For $r = -1$, $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\xrightarrow{rref}$ $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$
       eigenpair $-1 \leftrightarrow \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.
     - For $r = 3$, $\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$ $\xrightarrow{rref}$ $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$
       eigenpair $3 \leftrightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

   - **Form $x_h$.** A general solution of $x' = Ax$ is
     \[
     x_h = c_1e^{-t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
     \]

2. **Computation of $x_p$.**

   - **Rewrite $f$.** $f = \begin{bmatrix} -t - 1 \\ -4t - 2 \end{bmatrix}$ as $f = tk_1 + k_2$, where
     $k_1 = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$ and $k_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. Note that no element of $f$ occurs in the same element of a solution of the homogeneous equation. There is no interference.

   - **Accordingly, the form of $x_p$ is $x_p = ta + b$, where $a$ and $b$ are undetermined vector constants.**
(c) Substitute \( x = x_p \) into \( x' - Ax - f = 0 \) and collect like terms.

\[
\begin{align*}
    a - A(a + b) - tk_1 - k_2 &= 0 \\
    -a + A(a + b) + tk_1 + k_2 &= 0 \\
    t(Aa + k_1) + (Ab + k_2 - a) &= 0,
\end{align*}
\]

for all \( t \in \mathbb{R} \).

Thus \( A + k_1 = 0 \) implies \( a = -A^{-1}k_1 = A\backslash k_1 \).

Similarly, \( Ab + k_2 - a = 0 \) yields \( b = A\backslash(a - k_2) \).

Via MATLAB, we have \( a = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( b = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \).

Hence \( x_p = t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \).

3. Computation of a general solution \( x = x_p + x_0 \).

\[
\begin{align*}
x &= t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} + c_1 e^{t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
\end{align*}
\]

**MATLAB Examples**

**552/2 [revised]**

Find a general solution to the nonhomogeneous system \( x' = Ax + f \), where \( A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \) and \( f = \begin{bmatrix} -t - 1 \\ -4t - 2 \end{bmatrix} \).

**Solution**

Here we play fast and loose with symbols then creatively interpret “solutions” for \( a \) and \( b \). We’re pushing MATLAB beyond what it was intended to do... The diary file is rather long, so let’s first look at the script M-file driver.

```matlab
% NSS4-552/2
syms a b c1 c2 k1 k2 t
x = sym('x(t)');
A = sym([1 1; 4 1]);
% Homogeneous general solution
[V,D] = eig(A);
x1 = exp(-t) * [-1; 2];
x2 = exp(3*t) * [1; 2];
x = [x1, x2];
c = [c1; c2];
% Nonhomogeneous particular solution
x0 = diff(x,t) - A*x - (t*k1 + k2); xp = t*a + b;
eq0 = subs(deq0, a, b); eq0 = collect(eq0, t);
[a b] = solve(-A*a-k1, a-A*b-k2, a, b);% Nitty gritty
A = sym([1 1; 4 1]); k1 = [-1; -4]; k2 = [-1; -2]; a = A*k1 b = -A'*(-2) + (A*k2 + k1); xp = subs(xp);% Nonhomogeneous general solution
x = xp + xh; pretty(x);% Check check = diff(x,t) - A*x - (t*k1 + k2);
```

And here is the diary file with input and output interspersed.
% Homogeneous general solution
[V, D] = eig(A)

V =
[ 1, 1]
[-2, 2]

D =
[-1, 0]
[ 3, 3]
x1 = exp(-t) * [-1; 2];
x2 = exp(3*t) * [1; 2];
X = [x1, x2]

X =
[-exp(-t), exp(3*t)]
[2*exp(-t), 2*exp(3*t)]

c = [c1; c2]
c =
[ c1]
[ c2]

xh = X*c; pretty(xh)

% Nonhomogeneous particular solution
syms A % (Temporarily make A a symbol.)
de0 = diff(x, t) - A*x - (t*k1 + k2)
de0 =

diff(x(t), t) - A*x(t) - t*k1 - k2
xp = t*a + b;
eq0 = subs(de0, x, xp);
eq0 = collect(eq0, t)
eq0 =
(-A*a-k1)*t+a-A*b-k2
[a b] = solve(-a-b-k2, a, b);

% Nitty gritty
A = sym([2 2; 2 2])
I = sym(eye(2))
k1 = [-4; 0]
k2 = [-1; -2]
a = -A\k1
b = -A^(-2) * (A^*k2 + k1)
b =
[ 0]
[ 2]

xp = subs(xp)

% Nonhomogeneous general solution
x = xp + xh; pretty(x)

% Check
check = diff(x, t) - A*x - (t*k1 + k2)

check =
[ 0]
[ 0]

% NSS4-555/4
% Find a general solution to the nonhomogeneous system
x' = Ax + f, where A = [2 2; 2 2] and f = [-4*cos(t); -sin(t)].

Solution
First the script M-file driver.
% NSS4-555/4
% sym(A, b, a, c, 1, 2, t)
x = sym('x(t)');
A = sym([2 2; 2 2])

k1 = [-1; -4]
k1 =
[ -1]
[-4]
k2 = [-1; -2]
k2 =
[ -1]
[-2]
a = -A\k1
a =

b = -A^(-2) * (A^*k2 + k1)
b =
[ 0]
[ 2]
xp = subs(xp)

xp =
[ t]
[ 2]

% Nonhomogeneous particular solution
syms A % (Temporarily make A a symbol.)
de0 = diff(x, t) - A*x - (t*k1 + k2)
de0 =

diff(x(t), t) - A*x(t) - t*k1 - k2
xp = cos(t)*a + sin(t)*b;
eq0 = subs(de0, x, xp);
eq0 = collect(eq0, cos(t));
eq0 = collect(eq0, sin(t))
[a b] = solve(-a-b-k2, a, b);

% Nitty gritty
A = sym([2 2; 2 2])
I = sym(eye(2))
k1 = [-4; 0]
k2 = [0; -1]
a = - (I + A^2) \ (k2 + A*k1)
And now the diary file.

% % NSS4-555/4
% sym A a b c1 c2 k1 k2 t
x = sym('x(t)');
A = sym(eye(2));
% Homogeneous general solution
[V,D] = eig(A)
V =
[ -I, 1]
[ 1, 1]
D =
[ 0, 0]
[ 0, 4]
x1 = exp(0*t) * [-1; 1];
x2 = exp(4*t) * [1; 1];
X = [x1, x2]
X =
[ -1, exp(4*t)]
[ 1, exp(4*t)]
c = [c1; c2]
c =
[ c1]
[ c2]
xh = X*c; pretty(xh)
[-c1 + exp(4*t) c2]
[ c1 + exp(4*t) c2]
% Nonhomogeneous particular solution
sym A % (Temporarily make A a symbol.)
de0 = diff(x(t),t) - A*x - (cos(t)*k1 + sin(t)*k2)
de0 =
diff(x(t),t)-A*x(t)-cos(t)*k1-sin(t)*k2
xp = cos(t)*a + sin(t)*b;
eq0 = subs(de0, x, xp);
eq0 = collect(eq0, cos(t));
eq0 = collect(eq0, sin(t));
eq0 =
(-a-k2-A*b)*sin(t)+(A-a-k1+b)*cos(t)
[a b] = solve(-a-k2-A*b, -A*a-k1+b, a, b);
pretty(a), pretty(b)

% % Nitty gritty
A = sym(eye(2));
I =
[ 1, 0]
[ 0, 1]
k1 = [-4; 0]
k1 =
-4
k2 = [0; -1]
k2 =
0
-1
a = - (I + A^2) \ (k2 + A*k1)
a =
[ 0]
[ 1]
b = - (I + A^2) \ (A*k2 - k1)
b =
[-2]
[ 2]
kp = subs(xp)
kp =
[-2*sin(t)]
[ cos(t)+2*sin(t)]
% Nonhomogeneous general solution
x = kp + xh; pretty(x)
[-c1 + exp(4*t) c2]
[ c1 + exp(4*t) c2]
% Check
check = diff(x(t),t) - A*x - (cos(t)*k1 + sin(t)*k2)
check =
[ 0]
[ 0]
% echo off; diary off

555/26a

Find a general solution to the nonhomogeneous system
\[ x' = Ax + f, \] where \( A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \) and \( f = \begin{bmatrix} 3e^t \\ 6e^t \end{bmatrix} \).
Solution

We readily construct a general solution to the associated homogeneous system with MATLAB.

\[ x_h = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \]

Notice that components of \( f \) can occur as components of \( x_h \). (For example, take \( c_1 = 3 \) and \( c_2 = 0 \). Then \( 3 e^t \) occurs in the first element of \( x_h \).) Accordingly, we must adjust our first choice for \( x_p \) in the manner mentioned in the summary. Moreover, the fast-and-loose way we approached the first two problems won’t work. (There’s not enough structure inherent in our symbols.) Accordingly, we resort to the Full Monty: expressing all elements of vectors individually.

Therefore, let \( x_p = t e^t a + e^t b = t e^t \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + e^t \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \). This will work, but notice that there is not a unique solution to the linear system that we must solve after substituting into \( x' - Ax - f = 0 \). So we have some freedom as well as some ambiguity. This, together with the mathematical gymnastics involved in the method of undetermined coefficients, is why we’ll turn to the much more satisfying variation of parameters technique in the next lecture!

For now, peruse the needful in script M-file and diary formats as we construct our particular solution \( x_p = e^t \begin{bmatrix} 3 \\ 0 \end{bmatrix} \), then our general solution of the nonhomogeneous system \( x = x_p + x_h \).

(REMARK: The fact that \( a \) turned out to be the zero vector happens sometimes. Always make sure, however, to use the full form of the particular solution \( x_p \). For other choices of \( f \) it may well turn out that \( a \) is not the zero vector!)

```matlab
% delete p557x26a.txt; diary p557x26a.txt
clear; clc; close all; echo on
% % NSS4-557/26a
% sym s a b a1 a2 b1 b2 c1 c2 k t
x = sym('x(t)');
A = sym([0 1; -2 3])
% Homogeneous general solution
[V,D] = eig(A)
x1 = exp(t) * [1; 1];
x2 = exp(2*t) * [1; 2];
X = [x1, x2]
c = [c1; c2]
xh = X*c; pretty(xh)
% Nonhomogeneous particular solution
a = [a1;a2]; b = [b1;b2]; k = [3;6];
xp = t*exp(t)*a + exp(t)*b;
eq0 = diff(xp,t) - A*xp - exp(t)*k;
% All terms in the foregoing had a factor of exp(t).
% Since the output was VERY LONG, I suppressed it
% and divided out the exponential in the next command.
eq0 = simple(eq0 / exp(t));
eq0 = collect(eq0, t)
eq0 =
[ ...
(exp(t) c1 + exp(2 t) c2]
[ ...
(exp(t) c1 + 2 exp(2 t) c2]
% Nonhomogeneous particular solution
a = [a1;a2]; b = [b1;b2]; k = [3;6];
xp = t*exp(t)*a + exp(t)*b;
eq0 = diff(xp,t) - A*xp - exp(t)*k;
% All terms in the foregoing had a factor of exp(t).
% Since the output was VERY LONG, I suppressed it
% and divided out the exponential in the next command.
eq0 = simple(eq0 / exp(t));
eq0 = collect(eq0, t)
eq0 = ...
[ ...
(-a2+a1)*t+a1-3+b1-b2]
[ ...
(2*a1-2*a2)*t+a2-6-2*b2+2*b1]
[a1 a2 b1 b2] = solve( ...
-a2+a1, a1-3+b1-b2, 2*a1-2*a2, a2-6-2*b2+2*b1, ...
a1, a2, b1, b2)
a1 = 0
% %
```
\[ a_2 = 0 \]
\[ b_1 = 3 + b_2 \]
\[ b_2 = b_2 \]
\[ b_2 = 0 \]
\[ b_2 = 0 \]
\[ b_1 = \text{subs}(b_1) \]
\[ b_1 = 3 \]
\[ x_p = \text{subs}(x_p); x_p = \text{collect}(x_p, \exp(t)) \]
\[ x_p = \begin{bmatrix} 3 \exp(t) \\ 0 \end{bmatrix} \]
\[ \% \text{ A gen soln to NH lin sys} \]
\[ x = x_p + x_h; \text{pretty}(x) \]
\[ x = \begin{bmatrix} 3 \exp(t) \exp(t) c_1 + \exp(2 t) c_2 \\ \exp(t) c_1 + 2 \exp(2 t) c_2 \end{bmatrix} \]
\[ \% f = \exp(t) k; \]
\[ \text{check} = \text{simple}(\text{diff}(x, t) - A^x - f) \]
\[ \text{check} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
\[ \% \]
\[ \text{echo off; diary off} \]