Test C – Solutions (corrected)

Calculators may be used for simple arithmetic operations only!

1. (30 pts.) Let \( \vec{F} = (x - y)\hat{i} + z\hat{j} + (z - y)\hat{k} \).

(a) Calculate \( \nabla \cdot \vec{F} \).

\[
\frac{\partial (x - y)}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial (z - y)}{\partial z} = 1 + 0 + 1 = 2.
\]

(b) Calculate \( \nabla \times \vec{F} \).

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
(x - y) & z & (z - y)
\end{vmatrix} = \hat{i}(-1 - 1) + \hat{j}(0 - 0) + \hat{k}(0 + 1) = -2\hat{i} + \hat{k}.
\]

(c) Calculate \( \iint_R \vec{F} \cdot d\vec{S} \) when \( R \) is the portion of the plane \( z = 2 - 5x \) that lies in the quadrant \( z > 0 \), \( x > 0 \) and between the planes \( y = 0 \) and \( y = 1 \).

Note first that the equation of the surface can also be written as \( x = \frac{2 - z}{5} \), and that when \( x = 0 \), \( z = 2 \), and when \( z = 0 \), \( x = \frac{2}{5} \). Thus the integration will be from 0 to 2 in \( z \) or from 0 to \( \frac{2}{5} \) in \( x \). Now let’s write the integral as

\[
\iint [F_x \, dy \, dz + F_y \, dz \, dx + F_z \, dx \, dy]
\]

and ponder how best to integrate each term.

**Method 1:** Integrate each term over its own plane. The projection onto the \( x - z \) plane has zero area, so the \( F_y \) term is zero. The others are

\[
\int_0^1 dy \int_0^2 dz \,(x-y) + \int_0^1 dy \int_0^{2/5} dx \,(z-y) = \int_0^1 dy \int_0^2 dz \left(\frac{2-z}{5} - y\right) + \int_0^1 dy \int_0^{2/5} dx \,(2-5x-y)
\]

\[
= \int_0^2 dz \left[\frac{2-z}{5} - \frac{y^2}{2}\right]_0 + \int_0^1 dy \left[2x - \frac{5x^2}{2} - xy\right]_0^{2/5} = \int_0^2 dz \left[\frac{2-z}{5} - \frac{1}{2}\right] + \int_0^1 dy \left[\frac{4}{5} - \frac{2}{5} - \frac{2y}{5}\right] = \left[-\frac{z}{10} - \frac{z^2}{10}\right]_0 + \left[\frac{2y}{5} - \frac{y^2}{5}\right]_0 = -\frac{1}{5} - \frac{2}{5} + \frac{2}{5} - \frac{1}{5} = -\frac{2}{5}.
\]

**Method 2:** Integrate everything over \( x \) and \( y \). We have \( dz = -5 \, dx + 0 \, dy \). Therefore, the integral is

\[
\iint_D [(x-y) \, dy \,(−5 \, dx) + F_y \,(dx)^2 + (z-y) \, dx \, dy] = \int_0^1 dy \int_0^{2/5} dx \, [5x - 5y + 2 - 5x - y] + \int_0^1 dy \int_0^{2/5} dx \, [−6y + 2]
\]

\[
= \int_0^{2/5} dx \, [−3y^2 + 2y]_0^1 = \frac{2}{5}(-3 + 2) = -\frac{2}{5}.
\]
(d) Calculate \( \int_S \vec{F} \cdot d\vec{S} \) when \( S \) is the sphere \((x - 1)^2 + y^2 + (z + 2)^2 = 25\).

By Gauss’s theorem, this is the integral of \( \nabla \cdot \vec{F} \) over the ball whose boundary is \( S \). Since \( \nabla \cdot \vec{F} = 2 \), this is just twice the volume of the ball. Since the radius is 5, this is
\[
2 \cdot \frac{4\pi}{3} \cdot 5^3 = \frac{8\pi}{3} \cdot 125 = \frac{1000\pi}{3}.
\]

2. (10 pts.) Find the volume of the parallelepiped generated by the edges \( \vec{v}_1 = (1, 2, 1), \quad \vec{v}_2 = (2, 0, 2), \quad \vec{v}_3 = (1, 2, 3) \).

\[
\begin{vmatrix}
1 & 2 & 1 \\
2 & 0 & 2 \\
1 & 2 & 3 \\
\end{vmatrix} = -2 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -2(6 - 2) - 0 = -8.
\]

So the volume is +8.

3. (30 pts.) Define curvilinear coordinates \((u, t)\) by \( x = e^u \cosh t, \quad y = e^u \sinh t \).

(a) Find the formulas for the tangent vectors to the coordinate curves (at a generic point \((u, t)\)).

\[
\frac{\partial \vec{r}}{\partial u} = \left( e^u \cosh t, \quad e^u \sinh t \right), \quad \frac{\partial \vec{r}}{\partial t} = \left( e^u \sinh t, \quad e^u \cosh t \right).
\]

For future use, let us put these together (as columns) into the Jacobian matrix,
\[
J = \begin{pmatrix}
e^u \cosh t & e^u \sinh t \\
e^u \sinh t & e^u \cosh t
\end{pmatrix}.
\]

(b) Find the formulas for the normal vectors to the coordinate “surfaces” (which are actually curves in this two-dimensional case).

These are the rows of \( J^{-1} \). So we start with
\[
\det J = \begin{vmatrix} e^u \cosh t & e^u \sinh t \\ e^u \sinh t & e^u \cosh t \end{vmatrix} = e^{2u}(\cosh^2 t - \sinh^2 t) = e^{2u}.
\]

Therefore, by Cramer’s rule,
\[
J^{-1} = e^{-2u} \begin{pmatrix} e^u \sinh t & -e^u \sinh t \\ -e^u \cosh t & e^u \cosh t \end{pmatrix}.
\]

Thus
\[
\nabla u = \left( e^{-u} \cosh t, -e^{-u} \sinh t \right), \quad \nabla t = \left( -e^{-u} \sinh t, e^{-u} \cosh t \right).
\]

It is easy to check that these have the reciprocal orthonormality properties that they ought to have:
\[
\langle \frac{\partial \vec{r}}{\partial u}, \nabla u \rangle = 1, \quad \langle \frac{\partial \vec{r}}{\partial u}, \nabla t \rangle = 0, \quad \langle \frac{\partial \vec{r}}{\partial t}, \nabla u \rangle = 0, \quad \langle \frac{\partial \vec{r}}{\partial t}, \nabla t \rangle = 1.
\]
(c) Calculate $\int\int_D x^2y^2 \, dx \, dy$ when $D$ is the region bounded by the curves $u = 0, u = 2, t = 0, t = 1$.

$$\int_0^2 du \int_0^1 dt \, x^2y^2 J = \int_0^2 du \int_0^1 dt \, e^{5u} \cosh t \sinh^2 t$$

$$= \frac{1}{5} e^{5u} \frac{1}{3} \sinh^3 t \bigg|_0^1 = \frac{1}{15} (e^{10} - 1) \sinh^3 1.$$  

4. (15 pts.) Tell whether each of these formulas defines an inner product on the space $C(0, 5)$ (the real-valued continuous functions of $t$, where $0 < t < 5$). If not, briefly explain why not.

(a) $\langle f, g \rangle = \int_0^5 f(t)^2g(t)^2 \, dt$

NO — not bilinear.

(b) $\langle f, g \rangle = \int_0^5 \frac{f(t)g(t)}{1 + t^2} \, dt$

YES.

(c) $\langle f, g \rangle = \int_0^{\pi/2} f(t)g(t) \, dt$

NO — not positive definite: If $f(t) = 0$ for $t < \frac{\pi}{2}$ (but $f$ is not zero everywhere in the interval from $\frac{\pi}{2}$ to 5), then $\langle f, f \rangle = 0$ although $f$ is not the zero vector!

5. (15 pts.) Find an orthonormal basis for $\mathbb{R}^3$ whose first element is $\hat{u}_1 = \frac{1}{\sqrt{6}}(1, 1, 2)$. Note that $\hat{u}_1$ has norm one, so we can put it into the basis unchanged. We continue by the Gram–Schmidt procedure. Choose any vector linearly independent of $\hat{u}_1$ to be $\vec{v}_2$; for example, $\vec{v}_2 = (1, 0, 0)$. Its projection onto $\hat{u}_1$ is

$$\vec{v}_|| = (\hat{u}_1 \cdot \vec{v}_2)\hat{u}_1 = \frac{1}{6}(1 + 0 + 0)(1, 1, 2) = \frac{1}{6}(1, 1, 2).$$

So the perpendicular part is

$$\vec{v}_\perp = (1, 0, 0) - \frac{1}{6}(1, 1, 2) = \frac{1}{6}(5, -1, -2).$$

Since $\sqrt{25 + 1 + 4} = \sqrt{30}$, the normalized vector in this direction is

$$\hat{u}_2 = \frac{1}{\sqrt{30}}(5, -1, -2).$$

Now we need to find a unit vector perpendicular to the two we’ve found so far.

**Method 1:** $\hat{u}_3 \equiv \hat{u}_1 \times \hat{u}_2 =

\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 1 & 2 \\
5 & -1 & -2
\end{vmatrix}

= \frac{1}{\sqrt{6 \cdot 30}}(0 \hat{i} + 12 \hat{j} - 6 \hat{k}) = \frac{1}{\sqrt{5}}(0, 2, -1).$
Method 2: Let $\vec{v}_3 = (0, 1, 0)$. Its projection onto the plane of the first two vectors is

$$\vec{v}_\parallel = (\hat{u}_1 \cdot \vec{v}_3)\hat{u}_1 + (\hat{u}_2 \cdot \vec{v}_3)\hat{u}_2 = \frac{1}{6}(1, 1, 2) + \frac{-1}{30}(5, -1, -2).$$

So

$$\vec{v}_\perp = (\frac{-1}{6} + \frac{5}{30}, 1 - \frac{1}{6} - \frac{1}{30}, -\frac{2}{6} - \frac{-2}{30}) = (0, \frac{4}{5}, -\frac{2}{5})$$

which normalizes to the same $\hat{u}_3$ as before.