Test A – Solutions

Name: __________________________

Calculators may be used for simple arithmetic operations only!

1. (12 pts.) Find the inverse (if it exists) of the matrix \( M = \begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix} \).

Reduce the augmented matrix:

\[
\begin{pmatrix} 3 & 8 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \rightarrow [1] \rightarrow [2] \begin{pmatrix} 1 & 3 & 0 & 1 \\ 3 & 8 & 1 & 0 \end{pmatrix}
\]

\[
\rightarrow [2] \rightarrow [2] - 3[1] \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{pmatrix} \rightarrow [1] \rightarrow [1] + 3[2] \begin{pmatrix} 1 & 0 & 3 & -8 \\ 0 & 1 & -1 & 3 \end{pmatrix}.
\]

Therefore,

\[ M^{-1} = \begin{pmatrix} 3 & -8 \\ -1 & 3 \end{pmatrix}. \]

It is easy to check that \( MM^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

2. (10 pts.) A function \( f(x,y) \) satisfies the equations \( \frac{\partial f}{\partial x} = -\frac{2x}{y} f, \quad \frac{\partial f}{\partial y} = \frac{x^2}{y^2} f \). Calculate \( \frac{d}{dx} f(x, x^2) \).

\[
\frac{d}{dx} f(x, x^2) = \nabla f \cdot \frac{d}{dx} \begin{pmatrix} x \\ x^2 \end{pmatrix} = \left. \frac{\partial f}{\partial x} \right|_{y=x^2} \times 1 + \left. \frac{\partial f}{\partial y} \right|_{y=x^2} \times (2x)
\]

\[
= -\frac{2x}{x^2} f + \frac{x^2}{x^2} (2x) f = -2 + \frac{2x}{x} f = 0.
\]

Remark: \( f(x,y) = e^{-x^2/y} \) is a function with these properties.

3. (18 pts.) A curve \( C \) in three-dimensional space is specified by the parametric equations

\[ x = t, \quad y = t \sin t, \quad z = \cos t. \]

(a) Find the tangent vector to \( C \) at the point where \( t = \pi \).

Let \( \vec{r}(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \). Then \( \vec{r}'(t) = \begin{pmatrix} 1 \\ \sin t + t \cos t \\ -\sin t \end{pmatrix} \), so \( \vec{r}'(\pi) = \begin{pmatrix} 1 \\ -\pi \\ 0 \end{pmatrix} \).
(b) Find the directional derivative of \( f(x, y, z) = x + ze^y \) at that point, in the direction of the curve.

The unit vector in the direction of the curve is \( \vec{r}'(\pi) \) divided by its length:

\[
\hat{u} = \frac{1}{\sqrt{1 + \pi^2}} \begin{pmatrix} 1 \\ -\pi \\ 0 \end{pmatrix}. \quad \text{Also,} \quad \vec{r}(\pi) = \begin{pmatrix} \pi \\ 0 \\ -1 \end{pmatrix}.
\]

So

\[
\nabla f = (1, ze^y, e^y) \bigg|_{\vec{r}(\pi)} = (1, -1, 1).
\]

Thus

\[
\frac{\partial f}{\partial \hat{u}} = \nabla f \cdot \hat{u} = \frac{1 + \pi}{\sqrt{1 + \pi^2}}.
\]

4. (15 pts.) Producing a refrigerator requires 0.1 ton of steel and 0.2 ton of plastic. Producing an airplane requires 5 tons of steel and 2 tons of plastic. Producing a ton of steel consumes 3 tons of coal and 10 barrels of water. Producing a ton of plastic consumes 2 tons of coal and 50 barrels of water. Organize these facts into matrices, and find the matrix that tells you how much coal (\( c \)) and water (\( w \)) is needed to make \( r \) refrigerators and \( a \) airplanes.

Let \( s \) and \( p \) be the quantities of steel and plastic, and let

\[
\begin{pmatrix} s \\ p \end{pmatrix} = B \begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 0.1 & 5 \\ 0.2 & 2 \end{pmatrix} \begin{pmatrix} r \\ a \end{pmatrix}, \quad \begin{pmatrix} c \\ w \end{pmatrix} = A \begin{pmatrix} s \\ p \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 10 & 50 \end{pmatrix} \begin{pmatrix} s \\ p \end{pmatrix}.
\]

Then \( \begin{pmatrix} c \\ w \end{pmatrix} = AB \begin{pmatrix} r \\ a \end{pmatrix} \), where

\[
AB = \begin{pmatrix} 3 & 2 \\ 10 & 50 \end{pmatrix} \begin{pmatrix} 0.1 & 5 \\ 0.2 & 2 \end{pmatrix} = \begin{pmatrix} 0.7 & 19 \\ 11 & 150 \end{pmatrix}.
\]

5. (10 pts.) Classify each of these integral operators as linear, affine, or fully nonlinear (as a function of \( g \)). (\( g \) is an element of \( C(0, 1) \) — that is, a function.)

(a) \( A(g) = \int_0^t e^{(t-s)} g(s) \, ds \). (\( A(g) \) is another element of \( C(0, 1) \) — a function of the variable \( t \). In other words, \( A: C(0, 1) \to C(0, 1) \).)

**Linear.** This is clear from the form of the integrand; or, one can easily verify that

\[
A(\lambda g + h) = \int_0^t e^{(t-s)} [\lambda g(s) + h(s)] \, ds = \lambda \int_0^t e^{(t-s)} g(s) \, ds + \int_0^t e^{(t-s)} h(s) \, ds = \lambda A(g) + A(h).
\]
(b) \( B(g) = \int_0^1 te^{g(t)} \, dt \). (\( B(g) \) is an element of \( \mathbb{R} \) — a number. \( B: C(0,1) \rightarrow \mathbb{R} \).)

**Nonlinear.** Again, this is pretty obvious because the \( g \) is up in the exponent. A formal counterexample (to the homogeneity clause of the definition) is

\[
B(\lambda g) = \int_0^1 te^{\lambda g(t)} \, dt \neq \int_0^1 t\lambda e^{g(t)} \, dt = \lambda B(g).
\]

(\( B \) is not affine, because it’s not of the form of a linear operator plus a fixed vector (which would be a constant number in this case). An example of an affine operator \( C:C(0,1) \rightarrow C(0,1) \) is \( C(g)(t) = A(g)(t) + \cos t \), \( A \) as in part (a).)

6. (10 pts.) Construct the best affine approximation (also known as the first-order approximation) to \( T(x, y) = \left( \frac{\sqrt{x^2 + 4y^2}}{x - y} \right) \) in the neighborhood of the point \( (x_0, y_0) = (1, 1) \).

The matrix of partial derivatives is

\[
JT = \begin{pmatrix}
x & 4y \\
\sqrt{x^2 + 4y^2} & \sqrt{x^2 + 4y^2} -1
\end{pmatrix}, \quad \text{so} \quad J_{(1,0)} T = \begin{pmatrix}
\frac{1}{\sqrt{5}} & \frac{4}{\sqrt{5}} \\
0 & \frac{1}{\sqrt{5}} - \frac{4}{\sqrt{5}}
\end{pmatrix}.
\]

Therefore,

\[
T(x, y) \approx T(x_0, y_0) + J_{(1,0)} \left( x - x_0, y - y_0 \right) = \left( \frac{\sqrt{5}}{0} \right) + \left( \frac{1}{\sqrt{5}} \frac{4}{\sqrt{5}} \right) \left( x - 1, y - 1 \right).
\]

7. (25 pts.) Find all solutions \((x, y, z)\) of

\[
\begin{array}{c}
y - z = 1, \\
x - y - 2z = 0, \\
2x + 3y + Az = B,
\end{array}
\]

(A and B are arbitrary, but fixed, parameters. Certain special values of \( A \) and \( B \) will require special attention.)

\[
\begin{pmatrix}
0 & 1 & -1 & 1 \\
1 & -1 & -2 & 0 \\
2 & 3 & A & B
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & -1 & -2 & 0 \\
0 & 1 & -1 & 1 \\
2 & 3 & A & B
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & -3 & 1 \\
0 & 1 & -1 & 1 \\
0 & 0 & A + 4 & B
\end{pmatrix}.
\]

Case I: If \( A = -9 \) and \( B \neq 5 \), there are no solutions, because the bottom row gives an inconsistent equation.

Case II: If \( A = -9 \) and \( B = 5 \), then \( z \) is an arbitrary parameter and

\[
y = z + 1, \quad x = 3z + 1.
\]

Case III: If \( A \neq -9 \), continue reducing:

\[
\begin{pmatrix}
0 & 1 & 1 & 3 \\
0 & 1 & -1 & 1 \\
0 & 0 & 1 & C
\end{pmatrix}
\]

where we define \( C = \frac{B - 5}{A + 9} \) to save writing. Therefore, in this case there is the unique solution

\[
x = 1 + 3C, \quad y = 1 + C, \quad z = C.
\]