Test A – Solutions

**Calculators may be used for simple arithmetic operations only!**

1. (15 pts.) Find a parametric representation of the plane (in $\mathbb{R}^3$ with coordinates $(x,y,z)$) whose equation is $3x - 2z = -2$.

Let’s do this by the formal row-reduction method of Chapter 2. The augmented matrix and its reduced form are

\[
\begin{pmatrix}
3 & 0 & -2 & -2 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & -\frac{2}{3} & -\frac{2}{3} \\
\end{pmatrix}.
\]

Working from the end of the variable list back to the beginning, we see that we must take

\[z = t, \quad y = s \quad \text{(arbitrary parameters)},\]

and then

\[x = \frac{2}{3} t - \frac{2}{3}.
\]

In vector form,

\[
\vec{x} = s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{2}{3} \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{2}{3} \\ 0 \\ 0 \end{pmatrix}.
\]

It should be noted that there are other correct answers, such as

\[
\vec{x} = s \begin{pmatrix} 1 \\ 0 \\ \frac{3}{2} \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

Also, there are other ways of presenting an answer, such as

\[
\vec{x} = \begin{pmatrix} \frac{2}{3}(t-1) \\ s \\ t \end{pmatrix}.
\]

**Remarks:** The answer to a problem like this is easily checked: The constant term must by itself satisfy the equation, since it corresponds to $s = t = 0$. The vectors multiplied by $s$ and $t$ must satisfy the corresponding homogeneous equation, $3x - 2z = 0$. Unfortunately, passing these tests does not guarantee that your solution is complete: The main lesson of this problem is the necessity of including the term proportional to $(0,1,0)$ (or saying something about what the coordinate $y$ does!). If you leave out that term, you have constructed just a line, not a parametrized plane.
2. (15 pts.) An A2X30 module contains 1000 cubic inches of steel and 20 cubic inches of titanium. A B62W subassembly contains 10 cubic inches of steel and 1 cubic inch of titanium. A supertanker is built from 10 A2X30s and 8 B62Ws. A minesweeper is built from 5 A2X30s and 3 B62Ws.

Organize these facts into matrices, and find the matrix that should be used to calculate the quantities of metals needed to make \( k \) tankers and \( m \) minesweepers.

Let \( A, B, s, t \) have the obvious meanings. Translating the sentences into equations, we have

\[
\begin{pmatrix}
  s \\
  t
\end{pmatrix} = \begin{pmatrix} 1000 & 10 \\ 20 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{(let's call this matrix } M \text{)},
\]

\[
\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 10 & 5 \\ 8 & 3 \end{pmatrix} \begin{pmatrix} k \\ m \end{pmatrix} \quad \text{(let's call this matrix } N \text{)}.
\]

(Be sure that your matrices express the correct relationships, calculating the required inputs from the desired outputs. For example, the top line of the first matrix equation expresses the formula \( s = 1000A + 10B \), saying that we need 1000 units of steel for each A module and 10 units of steel for each B module.) Therefore,

\[
\begin{pmatrix}
  s \\
  t
\end{pmatrix} = MN \begin{pmatrix} k \\ m \end{pmatrix},
\]

where

\[
MN = \begin{pmatrix} 1000 & 10 \\ 20 & 1 \end{pmatrix} \begin{pmatrix} 10 & 5 \\ 8 & 3 \end{pmatrix} = \begin{pmatrix} 10080 & 5030 \\ 208 & 103 \end{pmatrix}.
\]

3. (20 pts.) Atoms near the point \( \vec{x}_0 \equiv \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \) sit in an electric field \( \vec{E} = \begin{pmatrix} x^2 - y \\ x^2 + y^2 \\ z^3 \end{pmatrix} \).

(a) Find the first-order (best affine) approximation to \( \vec{E}(\vec{x}) \) for \( \vec{x} \) near \( \vec{x}_0 \).

The Jacobian matrix is

\[
\frac{d\vec{E}}{d\vec{x}} = \begin{pmatrix}
  2x & -1 & 0 \\
  2x & 2y & 0 \\
  0 & 0 & 3z^2
\end{pmatrix} = \begin{pmatrix}
  6 & -1 & 0 \\
  6 & 4 & 0 \\
  0 & 0 & 3
\end{pmatrix} \text{ at } \vec{x}_0.
\]

Therefore,

\[
\vec{E}(\vec{x}) \approx \vec{E}(\vec{x}_0) + \frac{d\vec{E}}{d\vec{x}}(\vec{x} - \vec{x}_0) = \begin{pmatrix} 7 \\ 13 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 & -1 & 0 \\ 6 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x - 3 \\ y - 2 \\ z - 1 \end{pmatrix}.
\]
(b) Suppose that the index of refraction of a crystal depends on the electric field according to the law

\[ n = 1 + 0.01E_x^2 + 0.04E_y^2 + 0.02E_z^2. \]

Use the multidimensional chain rule to find \( \frac{\partial n}{\partial y} \) at \( \vec{x}_0 \).

**Method 1:**

\[ \nabla n = \frac{dn}{dE_x} \frac{d\vec{E}}{d\vec{x}} = (0.02E_x, 0.08E_y, 0.04E_z) \bigg|_{\vec{x}_0} \frac{d\vec{E}}{d\vec{x}} = (0.14, 1.04, 0.04) \begin{pmatrix} 6 & -1 & 0 \\ 6 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} = (\ast, 4.02, \ast), \]

where the numbers \( \ast \) are irrelevant and \( 4.02 = \frac{\partial n}{\partial y} \).

**Method 2:** (This is really the same method, but in “classical” partial-derivative notation instead of vectors and matrices.)

\[
\frac{\partial n}{\partial y} = \frac{\partial n}{\partial E_x} \frac{\partial E_x}{\partial y} + \frac{\partial n}{\partial E_y} \frac{\partial E_y}{\partial y} + \frac{\partial n}{\partial E_z} \frac{\partial E_z}{\partial y} = 0.02E_x (-1) + 0.08E_y (2y) + 0.04E_z (0).
\]

When all this is evaluated at \( (x, y, z) = (3, 2, 1) \), we again get 4.02.

4. (15 pts.) Let’s define a mapping \( Q \) of the function space \( C^1(0, \infty) \) into the function space \( C(0, \infty) \) by

\[ Q(f)(t) \equiv \frac{df}{dt} + (2t + 1)f(t)^2. \]

(Here \( t \) is the independent variable of the functions in \( C^1(0, \infty) \), and \( f \) is a generic element of \( C^1(0, \infty) \).) Is \( Q \) linear, affine, or nonlinear? Justify your answer.

Nonlinear. It is enough to show that either of the linearity conditions is violated; for instance,

\[ Q(2f) = 2 \frac{df}{dt} + 4(2t + 1)f^2 \neq 2 \frac{df}{dt} + 2(2t + 1)f^2 = 2Q(f). \]

5. (15 pts.)

(a) Solve the system \( \begin{cases} x + y = 2, \\ x - by = 0 \end{cases} \) for \( x \) and \( y \) (with \( b \) as a parameter).

Set up an augmented matrix and reduce:

\[
\begin{pmatrix} 1 & 1 & 2 \\ 1 & -b & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & -b - 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 - \frac{2}{b+1} \\ 0 & 1 & \frac{2}{b+1} \end{pmatrix}.
\]

We note that the last step assumes \( b + 1 \neq 0 \). Therefore, if \( b \neq -1 \), we have

\[ x = 2 - \frac{2}{b + 1} = \frac{2b}{b + 1}, \quad y = \frac{2}{b + 1}. \]
(b) Point out any values of $b$ that are “special” with regard to existence and uniqueness of solutions. (Explain.)

If $b = -1$, the equations are inconsistent; no solutions exist.

6. (20 pts.) Find the inverse (if it exists) of the matrix $M = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & -2 \\ 1 & 0 & 0 \end{pmatrix}$.

Therefore,

$$M^{-1} = \frac{1}{3} \begin{pmatrix} -2 & -1 & 5 \\ 4 & 2 & -7 \\ 1 & -1 & -1 \end{pmatrix}.$$