Test B – Solutions

1. (18 pts.) Determine whether each set is linearly independent. If it is not, find an independent set with the same span.

(a) \[ \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \\ -1 \\ 3 \\ 0 \\ 2 \\ 2 \\ 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \right\} \]

YES — obviously these two vectors are not multiples of each other. More formally,
\[
\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \\ 1 & 7 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & 2 \\ 0 & 5 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & \frac{9}{5} \\ 0 & 1 & -\frac{2}{5} \end{pmatrix},
\]
and we see that the number of nonzero rows has not decreased as we put the matrix into row echelon form.

(b) \[ \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 3 \\ 4 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \\ 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \]

Let’s put the vectors into a matrix as rows and reduce:
\[
\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 1 & 7 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & 2 \\ 0 & 5 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & \frac{11}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix}.
\]

So the answer is NO. A basis for the span is
\[ \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \\ \frac{1}{5} \end{pmatrix} \right\} \]
or any two of the original three vectors (since in this case none of the vectors is a multiple of any of the others).

2. (12 pts.) Two bases for the solution space of \( y'' = 9y \) are
\[ \mathcal{A} = \{ \bar{a}_1 = e^{3t}, \bar{a}_2 = e^{-3t} \} \quad \text{and} \quad \mathcal{B} = \{ \bar{b}_1 = \cosh(3t), \bar{b}_2 = \frac{1}{3} \sinh(3t) \}. \]

Find the matrix that expresses the coordinates of an arbitrary vector with respect to the \( \mathcal{A} \) basis in terms of its coordinates with respect to the \( \mathcal{B} \) basis.

From the definition of the hyperbolic functions, we have
\[
\bar{b}_1 = \frac{1}{3} \bar{a}_1 + \frac{1}{3} \bar{a}_2, \quad \bar{b}_2 = \frac{1}{3} \bar{a}_1 - \frac{1}{3} \bar{a}_2.
\]

From here there are many equivalent ways to proceed:
Method 1: Therefore, 
\[
\begin{pmatrix}
\frac{1}{7} & \frac{1}{12} \\
\frac{1}{6} & -\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{7} & \frac{1}{6} \\
\frac{1}{6} & -\frac{1}{6}
\end{pmatrix}
\]
maps the \(B\)-coordinates into the \(A\)-coordinates, as demanded.

Method 2: Therefore, 
\[
(\vec{b}_1, \vec{b}_2) = (\vec{a}_1, \vec{a}_2) \begin{pmatrix}
\frac{1}{7} & \frac{1}{6} \\
\frac{1}{6} & -\frac{1}{6}
\end{pmatrix}
\]
and the matrix appearing in this equation is the desired one.

Method 3: Therefore, if 
\[
y = x_1 \vec{a}_1 + x_2 \vec{a}_2 \quad \text{and also} \quad y = y_1 \vec{b}_1 + y_2 \vec{b}_2,
\]
then 
\[
y = y_1 \left( \frac{1}{2} \vec{a}_1 + \frac{1}{2} \vec{a}_2 \right) + y_2 \left( \frac{1}{6} \vec{a}_1 - \frac{1}{6} \vec{a}_2 \right) = \left( \frac{1}{2} y_1 + \frac{1}{6} y_2 \right) \vec{a}_1 + \left( \frac{1}{2} y_1 - \frac{1}{6} y_2 \right) \vec{a}_2.
\]

Therefore, 
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix}
\frac{1}{7} & \frac{1}{6} \\
\frac{1}{6} & -\frac{1}{6}
\end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.
\]

Remark: The factor \(\frac{1}{3}\) was not just thrown in to make the problem harder. \(\vec{b}_2\) is a “natural” solution to use, because it satisfies the initial conditions \(y(0) = 0\), \(y'(0) = 1\).

3. (32 pts.) A linear mapping \(L : \mathcal{P}_2 \rightarrow \mathcal{P}_2\) is defined by 
\[
[L(p)](t) = (t^2 - 4)p''(t) + tp'(t) - 4p(t) \quad \left( p' \equiv \frac{dp}{dt}, \text{etc.} \right).
\]

Note that \(L(t^2) = -8\) (more precisely: if \(p(t) = t^2\), then \(L(p)(t) = -8\) for all \(t\)). (This is free information! You don’t have to rederive it.)

(a) Find the matrix that represents \(L\) with respect to the standard basis \(\{t^2, t, 1\}\) for \(\mathcal{P}_2\).

\[
L(t^2) = -8, \quad L(t) = 0 + t - 4t = -3t, \quad L(1) = -4.
\]

Therefore, the matrix is 
\[
\begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ -8 & 0 & -4 \end{pmatrix}.
\]

(b) Is \(L\) surjective (onto \(\mathcal{P}_2\))? If not, what is its range?

NO. The range is \(\mathcal{P}_1\), the first-degree polynomials. This is clear either from the matrix — the span of the columns being the same as the span of 
\[
\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}
\]
— or from the action of \(L\) on the basis polynomials, which yields the constants and the multiples of \(t\) but no \(t^2\) terms.
(c) Is $L$ injective? If not, what is its kernel?

NO. From (b) and the dimension theorem, the kernel must have dimension $3 - 2 = 1$. To see what the kernel is, reduce the matrix:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 \\
-8 & 0 & -4 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & \frac{1}{2} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

Thus $p(t) = at^2 + bt + c$ is in the kernel if

$$a + \frac{1}{2}c = 0, \quad b = 0.$$

That is, the kernel consists of the multiples of the polynomial

$$-\frac{1}{2}t^2 + 1.$$

(d) Use “superposition principles” to find all solutions in $\mathcal{P}_2$ of the differential equation

$$(t^2 - 4)p''(t) + tp'(t) - 4p(t) = -16.$$

Obviously one solution is $p_p(t) = 2t^2$. The general solution, therefore, is $p = p_p + p_c$, where $p_c$ is the general element of the kernel (general solution of the corresponding homogeneous equation, $L(p) = 0$). Thus, all the solutions in $\mathcal{P}_2$ are

$$p(t) = 2t^2 + (-\frac{1}{2}t^2 + 1)c$$

for arbitrary numbers $c$.

(Of course, there are other solutions of the differential equation that are not quadratic polynomials.)

4. (22 pts.) The matrix $M = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \end{pmatrix}$ represents a linear operator $L: \mathbf{R}^3 \rightarrow \mathbf{R}^2$ with respect to the natural bases.

(a) What are the rank and the nullity of $M$? (The nullity is the dimension of the kernel of $L$.)

The rank is 2: Method 1: It can’t be greater than 2, because the dimension of the whole column space is just 2. It can’t be less than 2, because the three columns are not all proportional. Method 2: The (column) rank is equal to the row rank, and we already saw in Qu. 1(a) that the row rank of this matrix is 2.

Therefore, by the dimension theorem, the nullity is $3 - 2 = 1$. This can also be checked directly, by reducing the augmented matrix to (compare Qu. 1(a))

$$\begin{pmatrix}
1 & 0 & \frac{9}{5} & 0 \\
0 & 1 & -\frac{2}{5} & 0
\end{pmatrix},$$

from which it is clear that the third coordinate of the solution is arbitrary and the first and second coordinates are then determined.
(b) What matrix represents \( L \) if we switch to the basis \( \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \) for \( \mathbb{R}^2 \)? (The basis for the domain is still the natural one.)

Call the new basis vectors \( \vec{b}_1 \) and \( \vec{b}_2 \). In terms of the natural basis vectors, we have
\[
\vec{b}_1 = \hat{e}_1 + \hat{e}_2,
\]
\[
\vec{b}_2 = -\hat{e}_1 + \hat{e}_2.
\]
Therefore (reasoning as in Qu. 2),
\[
C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]
maps \( \vec{b} \)-coordinates into natural coordinates. (More quickly, this matrix is obtained simply by “stacking the new basis vectors together”.) Therefore,
\[
C^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}
\]
maps natural coordinates to \( \vec{b} \)-coordinates. To solve our problem we need to postprocess the natural-basis calculation of \( L \) with \( C^{-1} \). Therefore, the desired matrix is
\[
C^{-1}M = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & 1 & 5 \\ 1 & -3 & 3 \end{pmatrix}.
\]

5. (16 pts.)

(a) Explain what a “subspace” is.

A subspace of a vector space \( \mathcal{V} \) is a subset of \( \mathcal{V} \) that is closed under addition and scalar multiplication. The meaning of “closed” is made clear by the proofs below.

(b) Prove that one of these is a subspace (of a vector space \( \mathcal{V} \)). [Do either (A) or (B) – your choice.]

(A) The kernel of a linear function \( L: \mathcal{V} \rightarrow \mathcal{V} \).

Suppose that we have two elements of the kernel: \( L(\vec{v}_1) = 0 = L(\vec{v}_2) \). Then
\[
L(r\vec{v}_1 + \vec{v}_2) = rL(\vec{v}_1) + L(\vec{v}_2).
\]
That is, \( r\vec{v}_1 + \vec{v}_2 \) is also in the kernel. Thus the kernel is closed under the vector operations.

(B) The span of a list of vectors in \( \mathcal{V} \), \( \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N\} \).

Suppose that we have two elements of the span:
\[
\vec{u}_1 = \sum_{j=1}^{N} c_j \vec{v}_j \quad \text{and} \quad \vec{u}_2 = \sum_{j=1}^{N} d_j \vec{v}_j.
\]
Then
\[
r\vec{u}_1 + \vec{u}_2 = r \sum_{j=1}^{N} c_j \vec{v}_j + \sum_{j=1}^{N} d_j \vec{v}_j = \sum_{j=1}^{N} (rc_j + d_j)\vec{v}_j,
\]
which is also an element of the span, as was to be proved.