Example Sheet 10

1. If \( g \) is the inverse function of \( f(x) = \ln x + \tan^{-1} x \), find \( g'(\pi/4) \).

2. Study the second half of Page 268 in the text (the subsection entitled “The number \( e \) as a limit”). In particular, understand how the important formulae (6) and (7) on that page are derived.

3. Find the slope of the tangent to the curve \( y = x + \arctan y \) at the point \((1 - \pi/4, 1)\).

4. Evaluate:
   (i) \( \sin(2 \sin^{-1}(t)) \), \(-1 \leq t \leq 1\).
   (ii) \( \sin(2 \cos^{-1}(t)) \), \(-1 \leq t \leq 1\).
   (iii) \( \sin \left[ \sin^{-1}(u) + \sin^{-1}(v) \right] \), \(-1 \leq u, v \leq 1\).

5. (from one of my old final exams) Suppose that \( x \) is a number in the open interval \((\pi/2, 3\pi/2)\). What is \( \tan^{-1}(\tan x) \)?

6. Prove that for \( uv \neq 1 \),
   \[
   \tan^{-1} u + \tan^{-1} v = \tan^{-1} \left[ \frac{u + v}{1 - uv} \right],
   \]
   if the left-hand side (of the equation above) lies between \(-\pi/2\) and \(\pi/2\). (Recall and use the trigonometric identity
   \[
   \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.\]

7. Prove the following identities:
   (i) \( \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \)
   (ii) \( \cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y \)
   (iii) \( (\cosh x + \sinh x)^n = \cosh(nx) + \sinh(nx) \) for every real number \( x \) and every positive integer \( n \).

8. Let \( C \) denote the curve \( y = \cosh(x) \), \(-\infty < x < \infty\), and let \( \alpha \) be a fixed real number. Find the point on \( C \) where the slope of the tangent is \( \alpha \).

9. If \( x = \ln(\sec \theta + \tan \theta) \), show that \( \cosh x = \sec \theta \).