MATH 171-08c, Quiz 5

1. (8 marks) Let $c$ and $L$ be real numbers, and let $f$ be a function (defined in the appropriate interval(s)). Give formal definitions for each of the following:
   
   (i) $\lim_{x \to a^+} f(x) = -\infty$.
   
   (ii) $\lim_{x \to a^-} f(x) = L$.
   
   (iii) $\lim_{x \to a^+} f(x) = L$.
   
   (iv) $\lim_{x \to a^-} f(x) = -\infty$.

2. (9 marks) Determine the values of the numbers $a$ and $b$ such that
   
   $$\lim_{x \to a} \frac{2x^2 + ax + b}{x-a} = -1.$$

   Justify all your steps carefully and concisely.

3. (8 marks) Use the definition of differentiability to show that the function $f(x) = |x^2 - 1|$ is not differentiable at the points $-1$ and 1.

Solutions

1. (i) Given $T < 0$, there is a $\delta > 0$ such that $f(x) < T$ whenever $c < x < c + \delta$.

   (ii) Given $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $c - \delta < x < c$.

   (iii) Given $\epsilon > 0$, there is an $R < 0$ such that $|f(x) - L| < \epsilon$ whenever $x < R$.

   (iv) Given $T < 0$, there is an $R > 0$ such that $f(x) < T$ whenever $x > R$.

2. Let $N(x)$ and $D(x)$ denote the numerator and denominator, respectively, of the rational function $(2x^2 + ax + b)/(x - a)$. As $\lim_{x \to a} D(x) = 0$, $\lim_{x \to a} N(x)$ is finite (in fact, the limit is $3a^2 + b$), and $\lim_{x \to a} N(x)/D(x)$ is given to be finite, we must have $\lim_{x \to a} N(x) = 0$; that is, $3a^2 + b = 0$, which gives $b = -3a^2$. We use this fact in the following:

   $$-1 = \lim_{x \to a} \frac{2x^2 + ax + b}{x-a} = \lim_{x \to a} \frac{2x^2 + ax - 3a^2}{x-a} = \lim_{x \to a} \frac{(x-a)(2x+3a)}{x-a} = \lim_{x \to a} (2x + 3a) = 5a,$$

   where the third equation obtains from long division. Thus $a = -1/5$, whence $b = -3/25$.

3. We discuss the case $c = -1$; the other case is similar. Note that $f(-1) = 0$. Consider

   $$\lim_{x \to -1} \frac{f(x) - f(-1)}{x+1} = \lim_{x \to -1} \frac{|x^2 - 1|}{x+1} = \lim_{x \to -1} \frac{|x-1||x+1|}{x+1}.$$

   Now

   $$\frac{|x+1|}{x+1} = \begin{cases} \frac{-(x+1)}{x+1} = -1, & \text{if } x < -1; \\ \frac{x+1}{x+1} = 1, & \text{if } x > -1. \end{cases}$$

   Therefore

   $$\frac{|x-1||x+1|}{x+1} = \begin{cases} -|x-1|, & \text{if } x < -1; \\ |x-1|, & \text{if } x > -1, \end{cases}$$

   from which we obtain

   $$\lim_{x \to -1^-} \frac{|x-1||x+1|}{x+1} = -2 \quad \text{and} \quad \lim_{x \to -1^+} \frac{|x-1||x+1|}{x+1} = 2.$$

   It follows that $\lim_{x \to -1} \frac{f(x) - f(-1)}{x+1}$ does not exist; in other words, $f$ is not differentiable at $-1$. 

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