1. Suppose that $A$ is an $m \times n$ matrix whose rank is $n$. Show that if $x \neq 0$ and $y = Ax$, then $y \neq 0$.

2. Suppose that $A \in \mathbb{R}^{m \times n}$. Prove the following statements:
   (i) If $B$ is a nonsingular $m \times m$ matrix, then $N(BA) = N(A)$, and rank$(BA) = \text{rank}(A)$.
   (ii) If $C$ is a nonsingular $n \times n$ matrix, then rank$(AC) = \text{rank}(A)$.

3. Suppose that $A$ and $B$ are $n \times n$ matrices.
   (i) Prove that $AB = O_n$ if and only if $\text{CS}(B) \subseteq N(A)$.
   (ii) Assume that $AB = O_n$. Use (i) to show that $\text{rank}(A) + \text{rank}(B) \leq n$.

4. Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and let $x_0$ be a solution of the system $Ax = b$, that is, $Ax_0 = b$. Prove the following statements:
   (i) A vector $y \in \mathbb{R}^n$ is a solution of the system $Ax = b$ if and only if $y$ can be expressed in the form $y = x_0 + z$, for some $z \in N(A)$.
   (ii) If $N(A) = \{0\}$, then the solution $x_0$ is unique.

5. Suppose that $x$ and $y$ are nonzero vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and define the $m \times n$ matrix $A := xy^T$.
   (i) Show that $\{x\}$ is a basis for $\text{CS}(A)$, and that $\{y^T\}$ is a basis for $\text{RS}(A)$.
   (ii) Compute the dimension of $N(A)$.

6. Suppose that $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times r}$, and that $C := AB$. Prove the following statements:
   (i) $\text{CS}(C) \subseteq \text{CS}(A)$.
   (ii) $\text{RS}(C) \subseteq \text{RS}(B)$.
   (iii) rank$(C) \leq \text{min}\{\text{rank}(A), \text{rank}(B)\}$.
   (iv) If $A$ and $B$ both have linearly independent column vectors, then the column vectors of $C$ are also linearly independent.
   (v) Apply (iv) to $C^T$ and show that, if $A$ and $B$ have linearly independent row vectors, then the row vectors of $C$ are also linearly independent.
   (vi) If the columns of $B$ are linearly dependent, then the columns of $C$ are also linearly dependent.
   (vii) If the rows of $A$ are linearly dependent, then so are the rows of $C$.

**Definition.** Suppose that $A$ is an $m \times n$ matrix. We say that $A$ has a left inverse if there exists an $n \times m$ matrix $L$ such that $LA = I_n$. We say that $A$ has a right inverse if there exists an $n \times m$ matrix $R$ such that $AR = I_m$.

7. (i) Suppose that $A \in \mathbb{R}^{m \times n}$. Prove that the $A$ has a right inverse if and only if $\text{CS}(A) = \mathbb{R}^m$.
   (ii) Suppose that $A \in \mathbb{R}^{m \times n}$, and that $A$ has a right inverse. Use (i) to show that $m \leq n$.

8. Suppose that $A \in \mathbb{R}^{m \times n}$. Prove that the following statements are equivalent:
   (a) $A$ has a left inverse.
   (b) The columns of $A$ are linearly independent.