Example Sheet 7

**Definition.** A subset $S$ of the complex plane is said to be **convex** if for every pair of points $z_1, z_2 \in S$, the line segment joining $z_1$ and $z_2$ is contained entirely in $S$.

1. Which of the following sets is convex?
   
   (a) The upper-half plane
   (b) The closed unit disc $\overline{D}(0; 1)$
   (c) The open unit disc $D(0; 1)$
   (d) The unit circle $C(0; 1)$
   (e) The punctured plane $\mathbb{C} \setminus \{0\}$

2. Let $S$ be a nonempty subset of the complex plane which is open and convex. Suppose that $f$ is analytic at every point in $S$. Let $z_0$ be a fixed point in $S$. For any given $z \in S$, let $\gamma(z)$ be the straight line segment which starts at $z_0$ and terminates at $z$ (note that $\gamma(z) \subseteq S$ by convexity). Define $F(z) := \int_{\gamma(z)} f(\omega) d\omega$, $z \in S$. Prove that $F'(z) = f(z)$ for every $z \in S$.

3. Suppose that $S$ and $f$ are as in the preceding example. Prove that $\int_C f(z) dz = 0$ for every closed contour contained in $S$.

4. (i) Consider the integral $\int_0^{\infty} x^ne^{-x^{1/4}} \sin(x^{1/4}) dx$, where $n$ is a fixed nonnegative integer.
   
   (ii) Use the substitution $x = t^4$ to show that the integral can be transformed to $4 \int_0^{\infty} t^{4n+3} e^{-t} \sin t dt = 4I_n$, $n = 0, 1, 2, \ldots$.
   
   (iii) Let $R$ be a (large) positive number. Consider
   
   $$\int_{C_R} z^{4n+3} e^{(i-1)z} dz,$$
   
   where $C_R$ is the contour consisting of the following three pieces: the straight-line segment from $z = 0$ to $z = R$, followed by the quarter circle $\Gamma_R$ from $z = R$ to $z = iR$, followed by the straight-line segment from $z = iR$ to $z = 0$.
   
   (iv) Parametrize the first and third (straight-line) pieces of the contour $C_R$ to obtain the following:
   
   $$\int_0^R t^{4n+3} e^{-t}(e^{it} - e^{-it}) dt = -\int_{\Gamma_R} z^{4n+3} e^{(i-1)z} dz. \quad (1)$$
   
   (v) Show that the integral on the right of (1) approaches zero as $R$ tends to infinity.
   
   (vi) Conclude that $I_n = 0$ for every $n$.

5. (i) Define
   
   $$g(x) := \begin{cases} 
   x^3 \sin(\pi/x), & \text{if } 0 < x \leq 1; \\
   0, & \text{if } x = 0.
   \end{cases}$$
   
   Prove the following statements:
(a) $g$ is differentiable on $[0, 1]$ (with the usual understanding that the derivative at 0 stands for the right-derivative, whilst the derivative at 1 means the left-derivative).

(b) Show that $g'$ is continuous on $[0, 1]$ (as usual, continuity at the end points refers to one-sided continuity).

(ii) Let $C_1$ denote the curve parametrized by $z(t) = t + ig(t)$, $0 \leq t \leq 1$. Verify that $C_1$ is a smooth arc from $z = 0$ to $z = 1$.

(iii) Let $C$ denote the closed contour consisting of the path $C_1$ followed by the line segment $C_2$ along the real axis back to the origin. Verify that $C$ crosses itself at the points $z = 1/n$ for every positive integer $n$.

(iv) Let $f$ be an entire function, and let $C_3$ denote a smooth arc from the origin to the point $z = 1$; assume that $C_3$ does not intersect itself and has only its end points in common with the arcs $C_1$ and $C_2$. Show that

$$\int_{C_1} f(z) \, dz = \int_{C_3} f(z) \, dz \quad \text{and} \quad \int_{C_2} f(z) \, dz = -\int_{C_3} f(z) \, dz.$$ 

(v) Conclude that $\int_C f(z) \, dz = 0$, even though the closed contour $C$ has an infinite number of self-intersection points.

6. Suppose that $A$ is a fixed positive number. Establish the following slightly extended version of Jordan’s Lemma: $\int_0^\pi e^{-A \sin t} \, dt \leq \pi/A$.

The following example presents an alternate derivation of the inequality used in the proof of Jordan’s Lemma. It also has the advantage of establishing something more. You may attempt the example if you feel up to it; you may also ignore it should you so wish.

7. Define

$$f(x) = \begin{cases} 
\frac{\sin x}{x}, & \text{if } x \neq 0; \\
1, & \text{if } x = 0,
\end{cases}$$

and $g(x) = x \cos x - \sin x$, $x \in \mathbb{R}$.

(i) Observe that $f$ and $g$ are continuous on $[0, \pi]$.

(ii) Verify that $f'(x) = \frac{g(x)}{x^2}$, for $0 < x < \pi$.

(iii) Show that $g$ is decreasing on the interval $[0, \pi]$.

(iv) Use (iii) to show that $f$ is decreasing on $[0, \pi]$.

(v) Deduce that $\sin x \geq (2/\pi)x$ for $0 \leq x \leq \pi/2$. 