Concerning lower triangular matrices

1. Show that the product of any two lower triangular matrices is again lower triangular.

2. Suppose $L = (l_{ij})$ and $~L = (~l_{ij})$ are two $n \times n$ lower triangular matrices. Let $P$ be the product of $L$ and $~L$. Show that $P = (p_{ij})$ is an invertible lower triangular matrix. Let $P^{-1} = (p^{-1}_{ij})$. Deduce that the product of two unit-lower triangular matrices is also a unit-lower triangular matrix.

3. Suppose $L$ is an invertible $n \times n$ lower triangular matrix. Show that $L^{-1}$ is also lower triangular.

4. Suppose $L$ is an $n \times n$ lower triangular matrix with nonzero entries on the diagonal. Show that $L$ is invertible. Suppose $L$ is an invertible lower triangular matrix. Use Problem 2 above to show that $L$ has nonzero entries on its diagonal, and that the main-diagonal entries of $L^{-1}$ are the reciprocals of the corresponding main-diagonal entries of $L$. In particular, the inverse of a unit-lower triangular matrix is also unit lower triangular.

Concerning upper triangular matrices

5. Show that the product of any two upper triangular matrices is again upper triangular.

6. Suppose $U = (u_{ij})$ and $~U = (~u_{ij})$ are two $n \times n$ upper triangular matrices. Show that $U ~U = (u_{ij} ~u_{ij})$ is an invertible upper triangular matrix. Let $U^{-1} = (u^{-1}_{ij})$. Deduce that the product of two unit-upper triangular matrices is also a unit-upper triangular matrix.

7. Suppose $U$ is an invertible $n \times n$ upper triangular matrix. Show that $U^{-1}$ is also upper triangular.

8. Suppose $U$ is an $n \times n$ upper triangular matrix with nonzero entries on the diagonal. Show that $U$ is invertible. Suppose $U$ is an invertible upper triangular matrix. Use Problem 6 above to show that $U$ has nonzero entries on its diagonal, and that the main-diagonal entries of $U^{-1}$ are the reciprocals of the corresponding main-diagonal entries of $U$. In particular, the inverse of a unit-upper triangular matrix is also unit upper triangular.

Note: Results 5-8 for upper-triangular matrices may be deduced from the corresponding results for lower-triangular matrices using transposes.