**Norm:** Suppose $V$ is a (complex) vector space. A function $\| \cdot \| : V \to \mathbb{R}$ is said to be a *norm* on $V$ if it satisfies each of the following conditions:

[N1] $\|x\| \geq 0$ for every $x \in V$.
[N2] $\|x\| = 0$ if and only if $x = \mathbf{0}$.
[N3] $\|\alpha x\| = |\alpha| \|x\|$ for every $\alpha \in \mathbb{C}$ and every $x \in V$.
[N4] $\|x + y\| \leq \|x\| + \|y\|$ for every $x$ and $y$ in $V$.

**Inner Product:** Suppose $V$ is a (complex) vector space. A function $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C}$ is called an *inner product* on $V$ if it satisfies each of the following conditions:

[IP1] $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for every $x$ and $y$ in $V$.
[IP2] $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for every $x$, $y$, and $z$ in $V$.
[IP3] $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for every $\alpha \in \mathbb{C}$ and every $x$ and $y$ in $V$.
[IP4] $\langle x, x \rangle \geq 0$ for every $x \in V$, and $\langle x, x \rangle = 0$ if and only if $x = \mathbf{0}$.

**Orthogonal and Orthonormal Sets:** A subset $\{v_1, \ldots, v_n\}$ of an inner-product space $(V, \langle \cdot , \cdot \rangle)$ is said to be an *orthogonal set* if $\langle v_j, v_k \rangle = 0$ for every $j \neq k$. The set is called *orthonormal* if $\langle v_j, v_k \rangle = 0$ for every $j \neq k$ and $\langle v_j, v_j \rangle = 1$ for every $1 \leq j \leq n$.

**Orthogonal Projection:** Suppose $\{v_1, \ldots, v_n\}$ is an orthogonal set of nonzero vectors in an inner-product space $(V, \langle \cdot , \cdot \rangle)$. Let $x \in V$ and let $W := \text{span}\{v_1, \ldots, v_n\}$. The *orthogonal projection* of $x$ onto $W$ is defined as follows:

$$P_w(x) := \sum_{j=1}^{n} \frac{\langle x, v_j \rangle}{\langle v_j, v_j \rangle} v_j.$$ 

**Orthogonal Complement:** Suppose $W$ is a subspace of an inner-product space $(V, \langle \cdot , \cdot \rangle)$. The *orthogonal complement* of $W$ is defined as follows:

$$W^\perp := \{v \in V : \langle v, w \rangle = 0 \ \forall \ w \in W\}.$$ 

**Rank–Nullity Theorem:** Suppose $V$ and $W$ are vector spaces and $V$ is finite dimensional. If $L : V \to W$ is a linear transformation, then

$$\dim(V) = \dim(\mathcal{R}(L)) + \dim(\ker(L)),$$

where $\mathcal{R}(L)$ and $\ker(L)$ denote the range and kernel of $L$, respectively.

**Hölder’s Inequality:** Suppose $1 < p < \infty$ and $(1/p) + (1/q) = 1$. If $z_k, w_k \in \mathbb{C}$ for every $1 \leq k \leq n$, then

$$\sum_{k=1}^{n} |z_k w_k| \leq \left( \sum_{k=1}^{n} |z_k|^p \right)^{1/p} \left( \sum_{k=1}^{n} |w_k|^q \right)^{1/q}.$$
• **Minkowski’s Inequality:** Suppose $1 < p < \infty$. If $z_k, w_k \in \mathbb{C}$ for every $1 \leq k \leq n$, then
\[
\left( \sum_{k=1}^{n} |z_k + w_k|^p \right)^{1/p} \leq \left( \sum_{k=1}^{n} |z_k|^p \right)^{1/p} + \left( \sum_{k=1}^{n} |w_k|^p \right)^{1/p}.
\]

• **BCS Inequality:** If $(V, \langle \cdot, \cdot \rangle)$ is an inner-product space, then
\[
|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle}
\]
for every pair of vectors $x$ and $y$ in $V$.

• **Gram-Schmidt Theorem:** Suppose $\{v_1, \ldots, v_n\}$ is a subset of an inner-product space $(V, \langle \cdot, \cdot \rangle)$. Let $V_k := \text{span}\{v_1, \ldots, v_k\}$ for every $1 \leq k \leq n$. Define
\[
\mathbf{u}_k := \begin{cases} 
  v_1, & \text{if } k = 1; \\
  v_k - P_{k-1}(v_k), & \text{if } 2 \leq k \leq n,
\end{cases}
\]
where $P_{k-1}(v_k)$ denotes the orthogonal projection of $v_k$ onto the subspace $V_{k-1}$. The following hold:
(i) $\text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} = V_k$ for every $1 \leq k \leq n$.
(ii) The set $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthogonal set.
(iii) $\mathbf{u}_k = 0$, $2 \leq k \leq n$, if and only if $v_k \in V_{k-1}$.
Moreover, the $\mathbf{u}_k$’s can be found recursively as follows:
\[
\mathbf{u}_k = \begin{cases} 
  v_1, & \text{if } k = 1; \\
  v_k - \sum_{j=1}^{k-1} \alpha_{jk} \mathbf{u}_j, & \text{if } 2 \leq k \leq n,
\end{cases}
\]
where
\[
\alpha_{jk} := \begin{cases} 
  0, & \text{if } \mathbf{u}_j = 0; \\
  \frac{\langle v_k, \mathbf{u}_j \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle}, & \text{otherwise}.
\end{cases}
\]