Example Sheet 11

1. Suppose that \( \lambda_{\min} \) and \( \lambda_{\max} \) denote the smallest and largest eigenvalues, respectively, of an \( n \times n \) real symmetric matrix \( A \). Show that

\[
\lambda_{\min} = \min \left\{ \langle Ax, x \rangle : x \in \mathbb{R}^n \setminus \{0\} \right\} \quad \text{and} \quad \lambda_{\max} = \max \left\{ \langle Ax, x \rangle : x \in \mathbb{R}^n \setminus \{0\} \right\}.
\]

2. Suppose that \( A \) is an \( n \times n \) real symmetric matrix. Prove that the following are equivalent:
   (i) \( A \) is positive definite.
   (ii) \( \langle Ax, x \rangle > 0 \) for every nonzero vector \( x \in \mathbb{R}^n \).

3. Suppose \( A \) is an \( m \times n \) matrix.
   (i) Show that \( A^*A \) is positive semidefinite.
   (ii) Show that \( A^*A \) is positive definite if and only if the columns of \( A \) are linearly independent.
   (iii) Show that \( AA^* \) is positive semidefinite.
   (iv) Show that \( AA^* \) is positive definite if and only if the rows of \( A \) are linearly independent.

4. The following result is also known as Hadamard’s inequality: Let \( c_1, \ldots, c_n \) denote the columns of a nonsingular matrix \( C \in M_n \). Show that

\[
|\det(C)| \leq \prod_{k=1}^{n} \|c_k\|_2.
\]

(Consider \( C^*C \).)

5. (i) Peruse the proof of the AM-GM Inequality and show that equality holds therein if and only if all the numbers are equal (that is, the arithmetic mean of a set of nonnegative numbers equals the geometric mean of that set if and only if all the numbers are the same).
   (ii) Suppose that \( A \) is an \( n \times n \) positive definite matrix. Study the proof of Hadamard’s inequality (done in class); use (i) above to show that \( \det(A) = \prod_{k=1}^{n} [A]_{kk} \) if and only if \( A \) is diagonal.
   (iii) Show that equality obtains in Example 4 above if and only if the set \( \{c_1, \ldots, c_n\} \) is orthogonal.

6. Suppose that \( A \) is a positive definite matrix. Show that \( A \) can be factored in the form \( A = RR^* \) where the columns of \( R \) are orthogonal to each other. (Recall and use the fact that a positive definite matrix is unitarily diagonalizable.)

7. Suppose that \( A \) is a positive definite matrix in \( M_n \), and let \( \langle \cdot, \cdot \rangle \) denote the usual inner product in \( \mathbb{C}^n \). Define

\[
\langle x, y \rangle_A := \langle Ax, Ay \rangle, \quad x, y \in \mathbb{C}^n.
\]

Show that \( \langle \cdot, \cdot \rangle_A \) is an inner product in \( \mathbb{C}^n \).

8. (i) Let \( A, B \in M_n \). Assume that \( A \) is positive definite and \( B \) is positive semi-definite. Prove that

\[
\det(I_n + A^{-1}B) \geq 1 + \det(A^{-1}B).
\]
(ii) Deduce from (i) that
\[ \det(C_1 + C_2) \geq \det(C_1) + \det(C_2) \]
for every pair of positive semi-definite matrices \( C_1 \) and \( C_2 \).

(iii) Show by means of an example that assertion (ii) above is false, even if one of the two matrices in question is not positive semidefinite.

9. (posed by Mr. Wenyan He) Suppose that \( A \) is an \( m \times n \) matrix, and let \( B \) be the matrix obtained from \( A \) by deleting \( k \) rows. Show that \( \det(A^*A) \geq \det(B^*B) \). (Delete one row at a time.)

10. Suppose that \( A \in M_3(\mathbb{R}) \) is positive definite, and let \( \mathbf{x} \) denote a (variable) point \((x, y, z)\) in \( \mathbb{R}^3 \), as well as the vector \([x \ y \ z]^T\). Show that
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(\mathbf{x}^t A \mathbf{x})} \, dx \, dy \, dz = \frac{\pi \sqrt{\pi}}{\sqrt{\det(A)}}.
\]
(Remember and use the fact that \( A = QDQ^T \) for some real-orthogonal matrix \( Q \) and diagonal matrix \( D \). Make a suitable change of variables.)