Example Sheet 6

1. Suppose $X := \{x_1, \ldots, x_{n+1}\}$ is a set of distinct real numbers. Let $P_n$ denote the (real) vector space of all polynomials of degree at most $n$ with real coefficients; that is,

$$P_n := \left\{ f(x) = \sum_{k=0}^{n} a_k x^k : a_k \in \mathbb{R} \quad \forall \ 0 \leq k \leq n \right\}.$$  

(i) Suppose $f \in P_n$ and $f(x_j) = 0$ for every $1 \leq j \leq n+1$. Show that $f$ is identically zero. (Use Question 9 from Example Sheet 2.)

(ii) Let $X$ and $P_n$ be as above; define

$$\langle f, g \rangle_X := \sum_{j=1}^{n+1} f(x_j)g(x_j), \quad f, g \in P_n.$$  

Show that $\langle \cdot, \cdot \rangle_X$ is an inner product on $P_n$.

(iii) Recall that the set $B := \{1, x, x^2\}$ is the standard monomial basis for $P_2$. Let $X = \{-1, 0, 1\}$. Apply the Gram-Schmidt procedure to the set $B$ and obtain an orthogonal basis for $P_2$ with respect to the inner product $\langle \cdot, \cdot \rangle_X$.

2. A function $f$ from a normed linear space $(X, \| \cdot \|_X)$ into a normed linear space $(Y, \| \cdot \|_Y)$ is said to be Lipschitz if there exists a constant $M$ such that

$$\|f(u) - f(v)\|_Y \leq M \|u - v\|_X, \quad \forall u, v \in X.$$  

Prove that every linear transformation from $\ell^\infty_n(\mathbb{C})$ into itself is Lipschitz.

3. Suppose $(V, \langle \cdot, \cdot \rangle)$ is an inner product space, and let $L : V \rightarrow V$ be a linear transformation. The adjoint of $L$, denoted by $L^*$, is a map from $V$ to $V$ which is defined through the formula

$$\langle L^*(x), y \rangle = \langle x, L(y) \rangle, \quad x, y \in V.$$  

(i) Show that the adjoint of $L$ is unique.

(ii) Prove that $L^*$ is a linear transformation.

4. Let $V = \mathbb{C}^n$, equipped with the standard inner product. Let $L : V \rightarrow V$ be a linear transformation. Prove the following statement: If $A$ is the matrix of $L$ (with respect to the standard basis of $V$), then the matrix of $L^*$ (with respect to the standard basis of $V$) is $A^*$ (also denoted by $A^H$ in lecture).

5. Suppose $(V, \langle \cdot, \cdot \rangle)$ is a complex inner product space, and let $\| \cdot \|$ denote the norm induced by the inner product.

(i) Prove the polarization identity:

$$Re\langle x, y \rangle = \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 \right), \quad \forall x, y \in V.$$  

(ii) Show that

$$Re\langle x, y \rangle = \frac{1}{2} \left( \|x + y\|^2 - \|x\|^2 - \|y\|^2 \right), \quad \forall x, y \in V.$$
(iii) Prove:
\[ \| x + y \| \| x - y \| \leq \| x \|^2 + \| y \|^2, \quad \forall x, y \in V. \]

6. Let \( C(T) \) denote the set of all continuous, real-valued, 2\( \pi \)-periodic functions on the real line, that is, the set of all functions \( f : (-\infty, \infty) \to (-\infty, \infty) \) which are continuous on \( (-\infty, \infty) \), and satisfy the identity \( f(x + 2\pi) = f(x) \) for every real number \( x \).
   (i) Verify that \( C(T) \) is a subspace of (the familiar real vector space) \( C(-\infty, \infty) \).
Define
\[ \langle f, g \rangle := \int_{T} f(x)g(x) \, dx, \quad f, g \in C(T). \] (1)

(ii) Verify that \( \langle , \rangle \) given above is a (real) inner product on \( C(T) \).
Suppose \( n \geq 1 \) is a positive integer, and define
\[ S_n := \{ 1, \cos(x), \ldots, \cos(nx), \sin(x), \ldots, \sin(nx) \}, \]
and let
\[ T_n := \left\{ \alpha_0 + \sum_{k=1}^{n} [\alpha_k \cos(kx) + \beta_k \sin(kx)] : \alpha_k, \beta_k \in \mathbb{R} \right\}. \]
(iii) Verify that \( T_n \) is a subspace of \( C(T) \). (\( T_n \) is called the space of trigonometric polynomials of degree \( n \).)
(iv) Show that \( S_n \) is an orthogonal basis for \( T_n \) (the orthogonality here is with respect to the inner product defined in equation (1) above).
Given \( f \in C(T) \), define the Fourier coefficients of \( f \) as follows:
\[ a_k(f) := \frac{1}{\pi} \int_{0}^{2\pi} f(x) \cos(kx) \, dx, \quad 0 \leq k \leq n, \]
and
\[ b_k(f) := \frac{1}{\pi} \int_{0}^{2\pi} f(x) \sin(kx) \, dx, \quad 1 \leq k \leq n. \]
The \( n \)-th partial sum of the Fourier series of \( f \) is defined by the following equation:
\[ s_n(f, x) := \frac{a_0(f)}{2} + \sum_{k=1}^{n} [a_k(f) \cos(kx) + b_k(f) \sin(kx)], \quad n = 1, 2, \ldots. \]
(v) Show that \( s_n(f) \) enjoys the following best-approximation property:
\[ \| f - s_n(f) \| = \min_{\tau \in T_n} \| f - \tau \|, \]
where \( \| \cdot \| \) is the norm induced by the inner product in (1).

7. Suppose \( (V, \langle , \rangle) \) is a complex inner product space, and let \( y \) be a fixed vector in \( V \). Define
\[ L_y(x) := \langle x, y \rangle, \quad x \in V. \]
(i) Show that \( L_y \) is a linear functional on \( V \) (recall: a linear functional is a linear transformation from \( V \) to \( \mathbb{C} \)).
(ii) Consider $V$ as a normed linear space, equipped with the norm induced by the inner product. Show that the function $L_y$ is Lipschitz (recall the definition of a Lipschitz function from Example 2 above).

(iii) Suppose now that $V$ is finite dimensional. Let $L$ be a (generic) linear functional on $V$. Prove that there exists a vector $y$ in $V$ such that $L = L_y$; that is, there exists $y \in V$ such that $L(x) = \langle x, y \rangle$ for every $x \in V$.

Remark: Example 7(iii) above is the finite-dimensional version of the Riesz Representation Theorem for Hilbert spaces.