Infinite products

This section features a brief discussion of the basic theory of infinite products. Although this material is not part of the standard fare in introductory analysis courses, the reader will no doubt discern a strong connexion between the subject matter and the theory of infinite series (with which she is no doubt quite familiar).

We begin with a fundamental definition; owing to its rather tedious nature, it merits careful study.

**Definition 2.3.1.** Let \( \{u_n\} \) be a sequence of real numbers. We shall say the the infinite product \( \prod_{n=1}^{\infty} u_n \) converges to a number \( L \) if the following hold:

(a) There exists a positive integer \( p \) (usually taken to be the smallest such integer) such that \( u_n \neq 0 \) for every \( n \geq p \).

(b) With \( p \) as above, \( \lim_{q \to \infty} \prod_{n=p}^{q} u_n \in \mathbb{R} \setminus \{0\} \).

We denote

\[
L = \prod_{n=1}^{\infty} u_n := \prod_{n=1}^{p-1} u_n \prod_{n=p}^{q} u_n \lim_{q \to \infty} \prod_{n=p}^{q} u_n ,
\]

where the first product on the far right of (2.3.1) is defined to be 1 if \( p = 1 \).

In all other cases the infinite product \( \prod_{n=1}^{\infty} u_n \) is said to diverge.

In the theory of infinite products, an unavoidable distinction is made between a product which converges to zero and one which diverges to zero. Precisely one has the following:

**Remark 2.3.2.** (1) The infinite product \( \prod_{n=1}^{\infty} u_n \) diverges to 0 if one of the following conditions holds: (i) either \( u_n = 0 \) for infinitely many values of \( n \), or (ii) there is a positive integer \( p \) such that \( u_n \neq 0 \) for all \( n > p \), but \( \lim_{q \to \infty} \prod_{n=p}^{q} u_n = 0 \).

(2) The infinite product \( \prod_{n=1}^{\infty} u_n \) converges to 0 if the product converges (according to the stipulations of Definition 2.3.1) and \( u_n = 0 \) for at least one (but at most finitely many values of) \( n \).

**Example 2.3.3.** (1) The infinite product \( \prod_{n=1}^{\infty} [1 + (-1)^n] \) diverges to 0 because \( 1 + (-1)^n = 0 \) for infinitely many values of \( n \).

(2) Let \( u_n = (-1)^n \), \( n \in \mathbb{N} \). Then \( u_n \neq 0 \) for every \( n \), but \( \lim_{q \to \infty} \prod_{n=p}^{q} u_n = 0 \) does not exist for any \( p \). So the infinite product \( \prod_{n=1}^{\infty} u_n \) diverges.

(3) Let \( u_n = n/(n+1) \), \( n \in \mathbb{N} \). Then \( u_n \neq 0 \) for every \( n \), but for every positive integer \( p \) we have

\[
\lim_{q \to \infty} \prod_{n=p}^{q} u_n = \lim_{q \to \infty} \frac{p}{p+1} \frac{p+1}{p+2} \cdots \frac{q}{q+1} = \lim_{q \to \infty} \frac{p}{q+1} = 0 .
\]

Hence the infinite product \( \prod_{n=1}^{\infty} u_n \) diverges to 0.

(4) Let \( u_n = (n+1)/n \), \( n \in \mathbb{N} \). Then \( u_n \neq 0 \) for every \( n \), but for every positive integer \( p \) we have

\[
\lim_{q \to \infty} \prod_{n=1}^{q} u_n = \lim_{q \to \infty} \frac{p+1}{p} \frac{p+2}{p+1} \cdots \frac{q}{q} = \lim_{q \to \infty} \frac{q+1}{p} = +\infty.
\]

Hence the infinite product \( \prod_{n=1}^{\infty} u_n \) diverges to \(+\infty\).
(5) Let \( u_n = (n^2 - 1)/n^2 \), \( n \in \mathbb{N} \). Now \( u_1 = 0 \) and \( u_n \neq 0 \) for every \( n \geq 2 \). Moreover,
\[
\lim_{q \to \infty} \prod_{n=2}^{q} u_n = \lim_{q \to \infty} \prod_{n=2}^{q} \frac{(n-1)(n+1)}{n^2} = \lim_{q \to \infty} \left[ \prod_{n=2}^{q} \frac{n-1}{n} \prod_{n=2}^{q} \frac{n+1}{n} \right] = \lim_{q \to \infty} \left( \frac{1}{q} \right) \left( \frac{q+1}{2} \right) = \frac{1}{2}.
\]

Hence the infinite product \( \prod_{n=1}^{\infty} u_n \) converges to 0. (\( \prod_{n=1}^{\infty} u_n = u_1 \lim_{n \to \infty} \prod_{q=2}^{n} u_n = (0)(1/2) = 0 \).)

The analogy between this next result and Theorem 2.2.15 should be noted.

**Theorem 2.3.4.** (Cauchy Criterion for Infinite Products) The infinite product \( \prod_{n=1}^{\infty} u_n \) is convergent if and only if for every \( \epsilon > 0 \), there is some positive integer \( N = N(\epsilon) \) such that \( |\prod_{n=r}^{s} u_n - 1| < \epsilon \) whenever \( s - r \geq N \).

**Proof.** Suppose that \( \prod_{n=1}^{\infty} u_n \) is convergent, and let \( p \) be the smallest positive integer such that \( u_n \neq 0 \) for every \( n \geq p \), and \( \lim_{q \to \infty} \prod_{n=p}^{q} u_n =: \alpha \neq 0 \). The latter relation provides a positive integer \( R \), which may be chosen to be bigger than \( p \), such that \( |\prod_{n=p}^{q} u_n - \alpha| < |\alpha|/2 \) for every \( q > R \). Therefore \( |\prod_{n=p}^{s} u_n - |\alpha|| \leq |\alpha|/2 \) for every \( q > R \), whence \( |\prod_{n=p}^{s} u_n| \geq |\alpha|/2 \), \( q > R \). Furthermore, since \( u_n \neq 0 \) for \( p \leq n \leq R \), we conclude that there is a \( \Delta > 0 \) such that
\[
|\prod_{n=p}^{q} u_n| \geq \Delta \quad \text{for every } q \geq p. \tag{2.3.2}
\]

Since the sequence \( \{\prod_{n=p}^{q} u_n : q \geq p\} \) converges (as \( q \) tends to infinity), it is Cauchy (Theorem 2.2.13). So given \( \epsilon > 0 \), there is a positive integer \( N = N(\epsilon) \), which may be chosen to be larger than \( p \), such that \( |\prod_{n=p}^{s} u_n - \prod_{n=p}^{r-1} u_n| < \epsilon \Delta \) for every \( s \geq r \geq N \). So for such \( r \) and \( s \) we have
\[
\left| \prod_{n=r}^{s} u_n - 1 \right| = \left| \prod_{n=p}^{r-1} u_n \right| \left| \prod_{n=r}^{s} u_n - 1 \right| = \frac{\left| \prod_{n=p}^{r-1} u_n \prod_{n=r}^{s} u_n - 1 \right|}{\left| \prod_{n=p}^{r-1} u_n \right|} = \frac{\left| \prod_{n=p}^{r-1} u_n - \prod_{n=p}^{r-1} u_n \right|}{\left| \prod_{n=p}^{r-1} u_n \right|} < \frac{\epsilon \Delta}{\Delta} = \epsilon,
\]
the penultimate step resulting from (2.3.2). We have thus proved that the Cauchy criterion is necessary for convergence.

Conversely, let us assume that the Cauchy criterion is satisfied, and proceed to show that the infinite product \( \prod_{n=r}^{\infty} u_n \) is convergent. Firstly, the Cauchy criterion provides a positive integer \( p \) such that \( |\prod_{n=r}^{s} u_n - 1| < 1/2 \) for every \( s \geq r \geq p \). Consequently
\[
1/2 < \prod_{n=p}^{s} u_n < 3/2, \quad s \geq p; \tag{2.3.3}
\]
in particular \( u_n \neq 0 \) for every \( n \geq p \), and condition (a) of Definition 2.3.1 is fulfilled. It remains to show that \( \lim_{q \to \infty} \prod_{n=p}^{q} u_n \) exists and is nonzero. In fact, it suffices to demonstrate the former, for the latter will then follow automatically via (2.3.3). Thanks to Theorem 2.2.15, it is enough to show that the sequence \( \{\prod_{n=p}^{q} u_n : q \geq p\} \) is Cauchy. To that end, let \( \epsilon > 0 \) be arbitrary. By our assumption there is a positive integer \( N = N(\epsilon) \), which may be chosen to be larger than \( p \), such that
\[
\left| \prod_{n=r}^{s} u_n - 1 \right| < 2\epsilon/3, \quad s \geq r \geq N. \tag{2.3.4}
\]
If \( l > k > N, k, l \in \mathbb{N}, \) then (2.3.3) and (2.3.4) lead to

\[
\left| \prod_{n=p}^{l} u_n - \prod_{n=p}^{k} u_n \right| = \left| \prod_{n=p}^{k} u_n \right| \left| \prod_{n=k+1}^{l} u_n - 1 \right| < (3/2)(2\epsilon/3) = \epsilon.
\]

Thus \( \{\prod_{n=p}^{q} u_n : q \geq p\} \) is a Cauchy sequence and the proof is complete.

The following simple consequence of Theorem 2.3.4 should be compared with Remark 2.2.16(2).

**Corollary 2.3.5.** If \( \prod_{n=1}^{\infty} u_n \) is convergent, then \( \lim_{n \to \infty} u_n = 1 \).

**Proof.** Let \( \epsilon \) be an arbitrary positive number. The Cauchy criterion provides a positive integer \( N = N(\epsilon) \) such that \( |\prod_{n=r}^{s} u_n - 1| < \epsilon \) for every \( s \geq r \geq N \). Putting \( s = r \), one finds that \( |u_r - 1| < \epsilon \) for every \( s \geq N \), whence the result.

**Remark 2.3.6.** Example 2.3.3(4) demonstrates that the converse of Corollary 2.3.5 is false in general.

In what follows we shall adopt a slight change in notation and consider infinite products of the type \( \prod_{n=1}^{\infty} (1 + v_n) \); this will allow for a smoother formulation of the results that follow. The upcoming lemma will be of use in subsequent development.

**Lemma 2.3.7.** Suppose that \( a_n \geq 0 \) for every \( 1 \leq n \leq m \). The following hold:

(i) \( \prod_{n=1}^{m} (1 + a_n) \geq 1 + \sum_{n=1}^{m} a_n \).

(ii) If \( a_n < 1 \) for every \( 1 \leq n \leq m \), then \( \prod_{n=1}^{m} (1 - a_n) \geq 1 - \sum_{n=1}^{m} a_n \).

(iii) If \( a_n < 1 \) for every \( 1 \leq n \leq m \), then \( \prod_{n=1}^{m} (1 - a_n) \prod_{n=1}^{m} (1 + a_n) \leq 1 \).

**Proof.** (i) We use induction. The result being obvious for \( m = 1 \), let us assume that it holds for \( m = k \) and prove it for \( m = k + 1 \). Now

\[
\prod_{n=1}^{k+1} (1 + a_n) = \left[ \prod_{n=1}^{k} (1 + a_n) \right] (1 + a_{k+1}) \geq \left[ 1 + \sum_{n=1}^{k} a_n \right] (1 + a_{k+1})
\]

\[
= 1 + \sum_{n=1}^{k+1} a_n + a_{k+1} \sum_{n=1}^{k} a_n \geq 1 + \sum_{n=1}^{k+1} a_n,
\]

where the first inequality above comes from the induction hypothesis.

(ii) EXERCISE (Use induction.)

(iii) \( \prod_{n=1}^{m} (1 - a_n) \prod_{n=1}^{m} (1 + a_n) = \prod_{n=1}^{m} (1 - a_n^2) \leq 1 \).

This next result brings out some nice connexions between infinite series and products. In fact, the reader will find this to be a recurring refrain throughout the rest of this section.

**Theorem 2.3.8.** Suppose that \{\( v_n \)\} is a sequence of real numbers such that \( 0 \leq v_n < 1 \) for every positive integer \( n \). The following hold:
(i) If the infinite series \( \sum_{n=1}^{\infty} v_n \) is convergent, then the infinite product \( \prod_{n=1}^{\infty} (1 + v_n) \) is also convergent.

(ii) If \( \sum_{n=1}^{\infty} v_n \) is divergent, then \( \prod_{n=1}^{\infty} (1 + v_n) \) diverges to \(+\infty\).

(iii) If \( \sum_{n=1}^{\infty} v_n \) is convergent, then \( \prod_{n=1}^{\infty} (1 - v_n) \) is also convergent.

(iv) If \( \sum_{n=1}^{\infty} v_n \) is divergent, then \( \prod_{n=1}^{\infty} (1 - v_n) \) diverges to 0.

**Proof.** (i) Since \( 1 + v_n \geq 1 \) for every \( n \), it suffices to show that the sequence \( \{ \prod_{q=1}^{n} (1 + v_n) : q \in \mathbb{N} \} \) converges, as \( q \) tends to infinity, to a nonzero number. Let \( A_q := \prod_{q=1}^{n} (1 + v_n) \), \( q \in \mathbb{N} \), and observe that \( 1 \leq A_q \leq A_{q+1} \) for every \( q \). So it is enough to prove that the sequence \( \{ A_q : q \in \mathbb{N} \} \) is bounded above. The series \( \sum_{n=1}^{\infty} v_n \) being convergent, there is a positive integer \( R \) such that

\[
\sum_{n=R+1}^{\infty} v_n < 1/2. \tag{2.3.5}
\]

Lemma 2.3.7 and (2.3.5) imply the following for every \( q > R \):

\[
A_q = \left[ \prod_{n=1}^{R} (1 + v_n) \right] \left[ \prod_{q=R+1}^{n} (1 + v_n) \right] \leq \left[ \prod_{n=1}^{R} (1 + v_n) \right] \left[ \frac{1}{\prod_{n=R+1}^{q} (1 - v_n)} \right] \\
\leq \left[ \prod_{n=1}^{R} (1 + v_n) \right] \left[ \frac{1}{1 - \sum_{n=R+1}^{q} v_n} \right] \\
< 2 \prod_{n=1}^{R} (1 + v_n) < 2^{R+1}.
\]

Moreover, each of the numbers \( A_1, \ldots, A_R \) is no larger than \( 2^R \), so \( A_q < 2^{R+1} \) for every \( q \in \mathbb{N} \); the proof is complete.

(ii) Let \( A_q, q \in \mathbb{N} \), be as above. Lemma 2.3.7 asserts that \( A_q \geq 1 + \sum_{n=1}^{q} v_n \) for every \( q \). Since \( \lim_{q \to \infty} \sum_{n=1}^{q} v_n = +\infty \), it follows that \( \{ A_q \} \) diverges to \(+\infty\) as \( q \) tends to infinity.

(iii) Since \( 1 - v_n > 0 \) for every \( n \in \mathbb{N} \), it suffices to show that the sequence \( \{ B_q : q \in \mathbb{N} \} \), defined via \( B_q := \prod_{n=1}^{q} (1 - v_n) \), converges to a nonzero number as \( q \) tends to infinity. Now \( B_q \geq B_{q+1} \) for every \( q \in \mathbb{N} \), so it is enough to prove that \( \{ B_q \} \) is bounded below by a positive number. Let \( R \) be a positive integer satisfying (2.3.5). If \( q > R \), then Lemma 2.3.7 and (2.3.5) give rise to the relations

\[
B_q = \left[ \prod_{n=1}^{R} (1 - v_n) \right] \left[ \prod_{q=R+1}^{n} (1 - v_n) \right] \geq \left[ \prod_{n=1}^{R} (1 - v_n) \right] \left[ \frac{1}{\prod_{n=R+1}^{q} (1 - v_n)} \right] \\
> \frac{\prod_{n=1}^{R} (1 - v_n)}{2} = \frac{B_R}{2}.
\]

Since each of the numbers \( B_1, \ldots, B_R \) is positive, we find that the sequence \( \{ B_q \} \) is bounded below by a positive number.

(iv) Clearly \( 1 - v_n \) is positive for every \( n \); however, Lemma 2.3.7 provides the following:

\[
\prod_{n=1}^{q} (1 - v_n) \leq \frac{1}{\prod_{n=1}^{q} (1 + v_n)} \leq \frac{1}{1 + \sum_{n=1}^{q} v_n}, \quad q \in \mathbb{N}.
\]

The last term on the right converges to zero as \( q \) tends to infinity, because \( \lim_{q \to \infty} \sum_{n=1}^{q} v_n = +\infty \). Therefore the infinite product \( \prod_{n=1}^{\infty} (1 - v_n) \) diverges to zero.
Example 2.3.9. Suppose that $\alpha$ is a fixed real number. Verification of the following assertions is left to the reader:

(1) If $\alpha > 1$, then each of the infinite products $\prod_{n=1}^{\infty}(1 \pm n^{-\alpha})$ is convergent.

(2) If $0 < \alpha \leq 1$, then the infinite product $\prod_{n=1}^{\infty}(1 + n^{-\alpha})$ diverges to $\infty$, whilst the infinite product $\prod_{n=1}^{\infty}(1 - n^{-\alpha})$ diverges to zero.

Question: What happens when $\alpha \leq 0$?

Theorem 2.3.8 dealt with the case of infinite products in which the factors were all of the form $1 - v_n$ or $1 + v_n$, $0 \leq v_n < 1$. However, it remains silent about products of a ‘mixed’ type, for example: $\prod_{n=1}^{\infty}(1 + ((-1)^n/n))$. We shall now attempt to tackle some such cases; a pair of preludial lemmata is in order.

Lemma 2.3.10. If $-1/2 \leq x \leq 1/2$, then $0 \leq x - \ln(1 + x) \leq x^2$.

Proof. (i) We start with the first inequality. Let $f(x) := x - \ln(1 + x)$ and note that $\min\{f(x) : -1/2 \leq x \leq 1/2\} = f(0) = 0$.

(ii) The second inequality is proved as follows: Define $g(x) := x - \ln(1 + x) - x^2$ and observe that $\max\{g(x) : -1/2 \leq x \leq 1/2\} = g(0) = 0$.

Lemma 2.3.11. Suppose that $\{v_n\}$ is a square-summable sequence of real numbers, to wit, $\sum_{n=1}^{\infty} v_n^2$ is convergent. The following hold:

(i) There is a positive integer $p$ such that $|v_n| \leq 1/2$ for every $n \geq p$.

(ii) Let $p$ be as above and define $S_q := \sum_{n=p}^{q} (v_n - \ln(1 + v_n))$, $q \geq p$. Then the sequence $\{S_q : q \geq p\}$ converges as $q$ tends to infinity.

Proof. (i) Since $\sum_{n=1}^{\infty} v_n^2$ is convergent, $\lim_{n \to \infty} v_n^2 = 0$, so there is a $p \in \mathbb{N}$ such that $v_n^2 \leq 1/4$ for every $n \geq p$.

(ii) Lemma 2.3.10 ensures that $0 \leq v_n - \ln(1 + v_n) \leq v_n^2$ for every $n \geq p$. It follows that $\{S_q : q \geq p\}$ is a nondecreasing sequence bounded above by the (finite) number $\sum_{n=1}^{\infty} v_n^2$; its convergence is thus assured.

The following theorem will show, inter alia, that the infinite product $\prod_{n=1}^{\infty}(1 + ((-1)^n/n))$ (mentioned in the remark prefacing Lemma 2.3.10) is convergent.

Theorem 2.3.12. Suppose that $\sum_{n=1}^{\infty} v_n^2$ is convergent. The following hold:

(i) If the infinite series $\sum_{n=1}^{\infty} v_n$ is convergent, then so is the infinite product $\prod_{n=1}^{\infty}(1 + v_n)$.

(ii) If $\sum_{n=1}^{\infty} v_n = \infty$, then $\prod_{n=1}^{\infty}(1 + v_n)$ diverges to $+\infty$.

(iii) If $\sum_{n=1}^{\infty} v_n = -\infty$, then $\prod_{n=1}^{\infty}(1 + v_n)$ diverges to zero.

Proof. Let $p$ and $\{S_q\}$ be as in Lemma 2.3.11, and note that

$$\ln\left(\prod_{n=p}^{q} (1 + v_n)\right) = \sum_{n=p}^{q} \ln(1 + v_n) = \sum_{n=p}^{q} v_n - S_q, \quad q \geq p. \quad (2.3.6)$$

Now if $\lim_{q \to \infty} \sum_{n=p}^{q} v_n = A$ and $\lim_{q \to \infty} S_q = B$, then (2.3.6) implies the relation $\lim_{q \to \infty} \prod_{n=p}^{q} (1 + v_n) = e^{A-B} > 0$. Moreover, since $|v_n| \leq 1/2$ for every $n \geq p$, it is clear that $1 + v_n \neq 0$ for every such $n$. Thus assertion (i) is proven.

As for clause (ii), if $\sum_{n=1}^{\infty} v_n = +\infty$, then (2.3.6) shows that the sequence $\{\ln(\prod_{n=p}^{q} (1 + v_n))\}$ diverges to $+\infty$ as $q$ tends to infinity. Therefore $\lim_{q \to \infty} \prod_{n=p}^{q} (1 + v_n) = +\infty$, whence the required result.
Finally, if \( \sum_{n=1}^{\infty} v_n = -\infty \), then (2.3.6) shows that the sequence \( \{ \ln[\prod_{n=p}^{q}(1 + v_n)] \} \) also diverges to \(-\infty\) as \( q \) tends to infinity. Therefore \( \lim_{q \to \infty} \prod_{n=p}^{q}(1 + v_n) = 0 \) and the stated result follows.

Remark 2.3.13. The reader ought to be able to verify the following: if \( \alpha > 1/2 \) is a fixed number and \( v_n = (-1)^n/n^\alpha \) for every positive integer \( n \), then \( \prod_{n=1}^{\infty}(1 + v_n) \) is convergent.

We close this section with a brief discussion on absolute convergence of infinite products. The reader will do well to recall – from the vast recesses of her doubtless infallible memory – the corresponding notion for infinite series.

Definition 2.3.14. The infinite product \( \prod_{n=1}^{\infty}(1 + v_n) \) is said to be absolutely convergent if the product \( \prod_{n=1}^{\infty}(1 + |v_n|) \) is convergent.

Remark 2.3.15. Note that the infinite product \( \prod_{n=1}^{\infty} u_n \) is absolutely convergent if and only if the the product \( \prod_{n=1}^{\infty}(1 + |u_n - 1|) \) is convergent.

The following result, as well as the remark which will come after it, should come as no surprise.

Theorem 2.3.16. Every absolutely convergent infinite product is convergent.

Proof. Suppose that \( \prod_{n=1}^{\infty}(1 + v_n) \) is absolutely convergent, so that \( \prod_{n=1}^{\infty}(1 + |v_n|) \) is convergent. Let \( B_q := \prod_{n=1}^{q}(1 + |v_n|) \), \( q \in \mathbb{N} \), and observe that \( \{ B_q \} \) is a convergent sequence, hence bounded above. So Lemma 2.3.7(i) shows that the sequence \( \{ \sum_{n=1}^{q}|v_n| : q \in \mathbb{N} \} \) is bounded above, whence we may conclude that the series \( \sum_{n=1}^{\infty}|v_n| \) is convergent. Consequently \( \sum_{n=1}^{\infty} v_n \) is also convergent, and there is a positive integer \( p \) such that \( |v_n| \leq 1/2 \) for every \( n \geq p \). Now for every such \( n \) we also find that \( v_n^2 \leq |v_n| \). Therefore \( \{ v_n \} \) is a square-summable sequence and the required result follows from Theorem 2.3.12(i).

Remark 2.3.17. Let \( v_n := (-1)^n/n^\alpha \), \( n \in \mathbb{N} \), where \( 1/2 < \alpha \leq 1 \) is a fixed number. The reader will show that the infinite product \( \prod_{n=1}^{\infty}(1 + v_n) \) is convergent, but not absolutely convergent.