This next result, a celebrated one in the theory of Fourier series, is the centrepiece of this section.

**Theorem 4.6.14.** (Fejér) If \( F \in C(T) \), then

\[
\lim_{N \to \infty} \sigma_N[F; t] = F(t), \quad -\pi \leq t \leq \pi,
\]

the convergence being uniform on \([-\pi, \pi]\).

**Proof.** Let \( \Delta \) be the number given by Proposition 4.6.1(iii), so that \(|F(t)| \leq \Delta\) for every real number \( t \). Given \( \epsilon > 0 \), the second part of the aforementioned proposition supplies a positive number \( \delta \), depending on \( F \) and \( \epsilon \), such that

\[
|F(s) - F(t)| < \frac{\epsilon}{2} \quad \text{whenever } -2\pi \leq s, t \leq 2\pi \quad \text{and} \quad |s - t| < \delta.
\]

Choose a positive integer \( N_0 \) so large that

\[
\frac{\Delta}{\pi} \int_{\delta \leq |t| \leq \pi} |K_N(t)| \, dt < \frac{\epsilon}{2} \quad \text{for every } N \geq N_0;
\]

that such an \( N_0 \) exists is guaranteed by Proposition 4.6.13(iii).

Let \( t \) be any fixed (but arbitrarily chosen) point in \([-\pi, \pi]\), and let \( N \geq N_0 \). Proposition 4.6.11(ii), Theorem 4.6.8(ii), and Proposition 4.6.13(i) allow us to write

\[
\begin{aligned}
\sigma_N[F; t] - F(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(\tau) F(t - \tau) \, d\tau - \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(\tau) F(t) \, d\tau \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(\tau) [F(t - \tau) - F(t)] \, d\tau \\
&= \frac{1}{2\pi} \int_{|\tau| < \delta} K_N(\tau) [F(t - \tau) - F(t)] \, d\tau + \frac{1}{2\pi} \int_{\delta \leq |\tau| \leq \pi} K_N(\tau) [F(t - \tau) - F(t)] \, d\tau \\
&=: I_1 + I_2.
\end{aligned}
\]

As \( t - \tau \) and \( \tau \) lie in the interval \([-2\pi, 2\pi]\) whenever \( t, \tau \in [-\pi, \pi] \), and \(|t - \tau - t| = |\tau|\), we find, via (4.6.8), that \(|F(t - \tau) - F(t)| < \epsilon/2\) whenever \(|\tau| < \delta\). Combining this with Theorem 4.6.2(ii) and parts (i) and (ii) of Proposition 4.6.13, we obtain

\[
|I_1| \leq \frac{1}{2\pi} \int_{|\tau| < \delta} K_N(\tau) |F(t - \tau) - F(t)| \, d\tau < \frac{(\epsilon/2)}{2\pi} \int_{|\tau| < \delta} K_N(\tau) \, d\tau < \frac{(\epsilon/2)}{2\pi} \int_{-\pi}^{\pi} K_N(\tau) \, d\tau = \frac{\epsilon}{2}.
\]

As for \( I_2 \), the triangle inequality asserts that \(|F(t - \tau) - F(\tau)| \leq |F(t - \tau)| + |F(\tau)| \leq 2\Delta\) for every \( t \) and \( \tau \). Therefore Theorem 4.6.2(ii) implies that

\[
|I_2| \leq \frac{1}{2\pi} \int_{\delta \leq |\tau| \leq \pi} 2\Delta K_N(\tau) \, d\tau = \frac{\Delta}{\pi} \int_{\delta \leq |\tau| \leq \pi} K_N(\tau) \, d\tau < \frac{\epsilon}{2},
\]

the final inequality coming from (4.6.9).

Putting (4.6.11) and (4.6.12) together with (4.6.10), one infers that

\[
|\sigma_N[F; t] - F(t)| < \epsilon \quad \text{for every } N \geq N_0.
\]
Remembering that the choice of \( N_0 \) was independent of \( t \), we conclude that the estimate (4.6.13) holds uniformly in \( t \), and this completes the proof.

An immediate corollary of the preceding theorem is the following:

**Corollary 4.6.15.** Given \( F \in C(T) \) and \( \epsilon > 0 \), there is a trigonometric polynomial \( p \) such that 
\[
|F(t) - p(t)| < \epsilon \text{ for every } t \in [-\pi, \pi].
\]

**Proof.** Choose \( p = \sigma_N[F] \) for sufficiently large \( N \).

Fejér’s Theorem also allows us to deduce the following noteworthy facts about Fourier series of continuous functions.

**Corollary 4.6.16.** Suppose that \( F \in C(T) \).

(i) If \( \lim_{N \to \infty} S_N[F; t_0] = \alpha \) for some \( t_0 \in [-\pi, \pi] \), then \( \alpha = F(t_0) \).

(ii) If \( \lim_{N \to \infty} S_N[F; t] = 0 \) for every \( t \in [-\pi, \pi] \), then \( F[n] = 0 \) for every integer \( n \).

**Proof.** EXERCISE.

Three important consequences of Fejér’s Theorem will be presented next. We shall begin with a preludial lemma.

**Lemma 4.6.17.** Suppose that \( F \) and \( G \) belong to \( C(T) \), and let \( c_1 \) and \( c_2 \) be complex numbers. If \( H = c_1 F + c_2 G \), then \( H[n] = c_1 F[n] + c_2 G[n] \) for every integer \( n \).

**Proof.** EXERCISE.

The following result implies that a function in \( C(T) \) is completely determined by its sequence of Fourier coefficients.

**Theorem 4.6.18.** (Uniqueness Theorem) Suppose that \( F \) and \( G \) belong to \( C(T) \), and that \( F[n] = G[n] \) for every integer \( n \). Then \( F = G \).

**Proof.** Let \( H := F - G; \) so \( H \in C(T) \) and \( H[n] = 0 \) for every \( n \), the latter assertion being a consequence of Lemma 4.6.17. Therefore Proposition 4.6.11(ii) shows that \( \sigma_N[H; t] = 0 \) for every \( N \) and \( t \), so \( H = 0 \) by Fejér’s Theorem.

The forthcoming discussion will involve the following notions: a sequence \( \{ z_n : n \in \mathbb{Z} \} \) of complex numbers is said to be bounded if there is a number \( A \) such that \( |z_n| \leq A \) for every integer \( n \). We say that \( \lim_{|n| \to \infty} z_n = 0 \) if \( \lim_{|n| \to \infty} |z_n| = 0 \); quantitatively, given \( \epsilon > 0 \) there is a positive integer \( N \) such that \( |z_n| < \epsilon \) whenever \( |n| > N \). The reader will convince herself (preferably by supplying a proof) of the following: if \( \lim_{|n| \to \infty} z_n = 0 \), then \( \{ z_n : n \in \mathbb{Z} \} \) is a bounded sequence.

We now turn our attention briefly to the behaviour of the Fourier coefficients of a continuous function. Suppose that \( F \in C(T) \), and let \( \Delta := \sup \{|F(t)| : t \in \mathbb{R} \} \). If \( n \) is any integer, then Theorem 4.6.2(ii) and the fact that \( |e^{i\theta}| = 1 \) for every real number \( \theta \) imply the relations
\[
|F[n]| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t)e^{-int} \, dt \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(t)| \, dt \leq \frac{\Delta}{2\pi} \int_{-\pi}^{\pi} \, dt = \Delta;
\]
in particular, the sequence \( \{ F[n] : n \in \mathbb{Z} \} \) is bounded. Fejér’s Theorem allows us to establish a stronger result:
**Theorem 4.6.19.** *(Riemann–Lebesgue Lemma)* If $F \in C(T)$, then $\lim_{|n| \to \infty} F[n] = 0$.

**Proof.** Let $\epsilon > 0$ be given. Corollary 4.6.15 provides a trigonometric polynomial $p(t) = \sum_{k=-N}^{N} c_k e^{ikt}$ such that $|F(t) - p(t)| < \epsilon$ for every $t \in [-\pi, \pi]$. Let $n$ be an integer whose modulus exceeds $N$. Then $k - n$ is a nonzero integer for every integer $k$ between $-N$ and $N$, so

$$
\int_{-\pi}^{\pi} p(t)e^{-i\pi t} dt = \sum_{k=-N}^{N} c_k \int_{-\pi}^{\pi} e^{i(k-n)t} dt = 0
$$

by (4.6.7). Ergo

$$
|F[n]| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(t) - p(t)|e^{-i\pi t} |dt \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(t) - p(t)| dt \leq \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} dt \leq \epsilon.
$$

Our final task will be to use Fejér’s Theorem to give another proof of the Weierstraß Polynomial Approximation Theorem (Theorem 3.9.17). We begin by laying some groundwork.

**Lemma 4.6.20.** Suppose that $F \in C(T)$ is real valued and even. The following hold:

(i) Every Fourier coefficient of $F$ is a real number.

(ii) $F[-k] = F[k]$ for every positive integer $k$.

(iii) If $N$ is any nonnegative integer, then $\sigma_N[F; t]$ is a polynomial of the form $\sum_{k=0}^{N} A_k \cos(kt)$, where the $A_k$’s are real numbers.

**Proof.** (i) If $n = 0$, then

$$
F[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) dt \in \mathbb{R}.
$$

Suppose now that $n \neq 0$. Then

$$
F[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t)e^{-i\pi t} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) \cos(nt) dt - \frac{i}{2\pi} \int_{-\pi}^{\pi} F(t) \sin(nt) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) \cos(nt) dt \in \mathbb{R},
$$

the final equation stemming from the fact that the function $t \mapsto F(t) \sin(nt)$ is odd.

(ii) This is immediate from (4.6.14) because $\cos(-kt) = \cos(kt)$.

(iii) Recall from Proposition 4.6.11(ii) that $\sigma_N[F; t] = \sum_{k=-N}^{N} \alpha_k e^{ikt}$, where

$$
\alpha_k := \left(1 - \frac{|k|}{N + 1}\right) F[k], \quad -N \leq k \leq N.
$$

So assertions (i) and (ii) of the current lemma imply that $\alpha_k$ is a real number for every $|k| \leq N$ and $\alpha_{-k} = \alpha_k$ for every positive integer $k$. Therefore

$$
\sigma_N[F; t] = \alpha_0 + \sum_{|k|=1}^{N} \alpha_k \cos(kt) + i \sum_{k=1}^{N} \alpha_k \sin(kt)
$$

$$
= \alpha_0 + \sum_{k=1}^{N} (\alpha_k + \alpha_{-k}) \cos(kt) + i \sum_{k=1}^{N} (\alpha_k - \alpha_{-k}) \sin(kt)
$$

$$
= \alpha_0 + 2 \sum_{k=1}^{N} \alpha_k \cos(kt).
$$
Our preparations are completed with the following lemma, which asserts that every trigonometric polynomial of the type encountered in the third part of the previous result is expressible as a linear combination of powers of the cosine function.

Lemma 4.6.21. Every trigonometric polynomial of the form

\[ p(t) = \sum_{k=0}^{N} A_k \cos(kt), \quad A_k \in \mathbb{R}, \quad 0 \leq k \leq N, \]

admits the following representation:

\[ p(t) = \sum_{k=0}^{N} B_k \cos^k t, \quad B_k \in \mathbb{R}, \quad 0 \leq k \leq N. \]

Proof. It suffices to show that, for every positive integer \( m \), the function \( \cos(mt) \) is a real linear combination of the set \( \{ \cos^l t : 0 \leq l \leq m \} \). The DeMoivre and Binomial Theorems combine to yield the identities

\[
\cos(mt) + i\sin(mt) = (\cos t + i\sin t)^m = \sum_{k=0}^{m} \binom{m}{k} i^k \sin^k t \cos^{m-k} t, \quad t \in \mathbb{R}.
\]

Comparing the real parts of the first and third terms above, one finds that

\[
\cos(mt) = \sum_{r=0}^{[m/2]} \binom{m}{2r} i^{2r} \sin^{2r} t \cos^{m-2r} t = \sum_{r=0}^{[m/2]} \binom{m}{2r} (-1)^r (1 - \cos^2 t)^r \cos^{m-2r} t, \quad t \in \mathbb{R},
\]

and the last expression is a real linear combination of the functions \( 1, \cos t, \ldots, \cos^m t \).

We are now ready for Weierstraß's theorem, which will be given in two steps. The first (and key) step is the following:

Theorem 4.6.22. (Weierstraß) Suppose that \( g : [-1,1] \to \mathbb{R} \) is continuous on \([-1,1]\). Given \( \epsilon > 0 \), there is a polynomial \( s(x) = \sum_{k=0}^{N} B_k x^k, B_k \in \mathbb{R}, 0 \leq k \leq N, \) such that \( |(g(x) - s(x))| < \epsilon \) for every \( x \in [-1,1] \).

Proof. Define \( F(t) := g(\cos t) \), which is a real-valued, even function belonging to \( C(T) \). Thanks to Fejér’s Theorem, Lemma 4.6.20(iii), and Lemma 4.6.21, there is a nonnegative integer \( N \) and real numbers \( B_0, \ldots, B_N \) such that

\[
\left| g(\cos t) - \sum_{k=0}^{N} B_k \cos^k t \right| = \left| F(t) - \sum_{k=0}^{N} B_k \cos^k t \right| < \epsilon \quad \text{for every } t \in [0, \pi]. \tag{4.6.15}
\]

Putting \( x = \cos t \) and \( s(x) = \sum_{k=0}^{N} B_k x^k \), we find that (4.6.15) is equivalent to

\[ |g(x) - s(x)| < \epsilon \quad \text{for every } x \in [-1,1]. \]
The general form of Weierstraß’s Theorem is derived from Theorem 4.6.22 in much the same way as Theorem 3.9.17 was deduced from Theorem 3.9.15.

**Theorem 4.6.23.** (Weierstraß’s Polynomial Approximation Theorem) Let \( a \) and \( b \) be a pair of fixed real numbers with \( a < b \). Suppose that \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\). Given \( \epsilon > 0 \), there is a polynomial \( P \) such that \(|f(y) - P(y)| < \epsilon\) for every \( y \in [a, b]\).

**Proof.** Define \( g(x) = f \left( a + \frac{(b-a)(x+1)}{2} \right) \), \(-1 \leq x \leq 1\). Then \( g \) is a continuous real-valued function on the interval \([-1,1]\). Accordingly, Theorem 4.6.22 supplies a polynomial \( s \) such that

\[
|f \left( a + \frac{(b-a)(x+1)}{2} \right) - s(x)| = |g(x) - s(x)| < \epsilon \quad \text{for every } x \in [-1,1]. \tag{4.6.16}
\]

Putting \( y = a + \frac{(b-a)(x+1)}{2} \) and \( P(y) := s \left( \frac{2(y-a)}{b-a} - 1 \right) \), one finds that (4.6.16) may be rewritten as

\[
|F(y) - P(y)| < \epsilon \quad \text{for every } y \in [a, b].
\]

As \( P \) is the composition of the polynomial \( s \) with the linear function \( y \mapsto \frac{2(y-a)}{b-a} - 1 \), \( P \) itself is a polynomial in the variable \( y \).