Lecture for Week 1 (Secs. 1.1 and 1.2)

Vectors and the Dot Product

NOTE: The review material on Stewart pp. 2–27 is not on our syllabus, because that might give you the misimpression that the course is going to be

(a) easy,

(b) boring.

So it's your responsibility to review it if you need to. (But remember **RULE 4**.)

Likewise, I will not repeat all the definitions about vectors.

What most students want in lectures is example problems, so that's what I try to provide. But to give (or understand) an example, one must first have something to exemplify, so remember **RULE 3**!

So I assume you have read at least Sec. 1.1 (and Sec. 1.2 by next class).

Let's start with an easy one to review the definitions.

Exercise 1.1.9 (p. 53)

$$\mathbf{a} = \langle 5, -12 \rangle, \quad \mathbf{b} = \langle -2, 8 \rangle.$$

Find $|\mathbf{a}|$, $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $2\mathbf{a}$, $3\mathbf{a} + 4\mathbf{b}$.

Always think about the problem yourself before going to the next frame!

$$\mathbf{a} = \langle 5, -12 \rangle, \quad \mathbf{b} = \langle -2, 8 \rangle.$$

$$|\mathbf{a}| \equiv \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13.$$

$$\mathbf{a} + \mathbf{b} \equiv \langle 5 + (-2), -12 + 8 \rangle = \langle 3, -4 \rangle.$$

$$\mathbf{a} - \mathbf{b} \equiv \langle 5 - (-2), -12 - 8 \rangle = \langle 7, -20 \rangle.$$

$$2\mathbf{a} \equiv \langle 2(5), 2(-12) \rangle = \langle 10, -24 \rangle.$$

Let's draw some pictures of these while we're at it. . . .

Remark: Vectors are also written this way:

$$\mathbf{a} = 5\mathbf{i} - 12\mathbf{j}, \quad \mathbf{b} = -2\mathbf{i} + 8\mathbf{j}.$$

So (to finish the exercise)

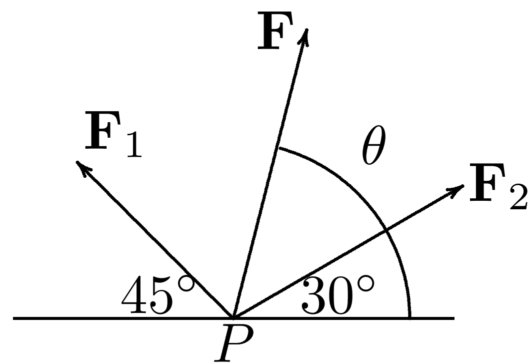
$$\begin{aligned} 3\mathbf{a} + 4\mathbf{b} &\equiv 3(5\mathbf{i} - 12\mathbf{j}) + 4(-2\mathbf{i} + 8\mathbf{j}) \\ &= (15 - 8)\mathbf{i} + (-36 + 32)\mathbf{j} \\ &= 7\mathbf{i} - 4\mathbf{j} \quad \text{or} \quad \langle 7, -4 \rangle. \end{aligned}$$

In this respect vectors act much like numbers, algebraically. ■

Now something more intimidating.

Exercise 1.1.27

Two forces with $|\mathbf{F}_1| = 10$ lb and $|\mathbf{F}_2| = 12$ lb act on a body at P. Find the resultant force $\mathbf{F}$ and its magnitude and direction.



Let's break the vectors into components.

$$\mathbf{F}_1 = \langle -10/\sqrt{2}, 10/\sqrt{2} \rangle \approx \langle -7.07, 7.07 \rangle,$$

$$\mathbf{F}_2 = \langle 12\sqrt{3}/2, 6 \rangle \approx \langle 10.39, 6.00 \rangle.$$

$$\mathbf{F} \approx \langle -7.07 + 10.39, 7.07 + 6.00 \rangle = \langle 3.32, 13.07 \rangle.$$

$$|\mathbf{F}| \approx \sqrt{3.32^2 + 13.07^2} \approx 13.5 \text{ lb},$$

$$\theta \approx \tan^{-1} \frac{13.07}{3.32} \approx \tan^{-1} 1.32 \approx 75.7^\circ.$$

I did the intermediate arithmetic in *Maple*. You can see my *Maple* worksheet at `forces.pdf` (in `http://calclab.math.tamu.edu/ fulling/o151`) .

Notice that I forgot how to convert radians to degrees, so I asked *Maple* for help, which showed up in a separate window. You don't get to profit from my other mistakes, because I corrected the commands and reexecuted them.

I went numerical early, because the sum of square roots (in the book's answer) is not illuminating. In general you should stick with exact answers as long as possible.

The more “realistic” a problem is, the more likely you are to need calculator or computer assistance. Some of the automated homework problems with randomly generated numbers can be tough (arithmetically), so come prepared with a calculator that handles trig functions.

On the other hand, you should not be using a calculator or *Maple* to evaluate things like $\cos(60^\circ)$ or to evaluate the derivatives and integrals you are supposed to be learning. Be warned that calculators will probably be **banned** from exams. ■

On to the dot product and projections.

Exercise 1.2.13

Find the angle (in degrees) between $\mathbf{a} = \langle 1, 2 \rangle$ and $\mathbf{b} = \langle 3, 4 \rangle$.

The two key formulas for the dot product are

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

and

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 \,.$$

Note, by the way, that $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}\,.$

Use the second formula to find $\mathbf{a} \cdot \mathbf{b}$ and then use the first formula to find θ:

$$\mathbf{a} = \langle 1, 2 \rangle, \quad \mathbf{b} = \langle 3, 4 \rangle.$$

$$1 \times 3 + 2 \times 4 = \sqrt{1^1 + 2^2}\sqrt{3^2 + 4^2} \cos\theta\,;$$

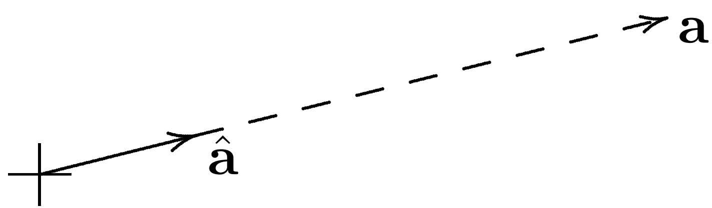
$$\theta = \cos^{-1} \frac{11}{\sqrt{5}\sqrt{25}} = \cos^{-1} \frac{11}{5\sqrt{5}} \approx 10^\circ$$

(where I trust the book's answer key for the arithmetic in the final step). ■

I intend to take a slightly different approach (from the book's) to projections.

Definition: Given a nonzero vector $\mathbf{a}$, the **unit vector in the direction of a** is

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}\,.$$



Note that $|\hat{\mathbf{a}}| = 1$. "Unit vector" just means "vector with length 1". Unit vectors are usually printed with caret (hat) accents. In many books the basic unit vectors $\mathbf{i}$ and $\mathbf{j}$ also have carets, but Stewart and I don't write them that way, because the carets get in the way of the dots or vice versa.

There is another unit vector parallel to $\mathbf{a}$, namely $-\hat{\mathbf{a}}$.

Exercise 1.2.35

Find the scalar projection and the vector projection of $\mathbf{b} = \langle 1, 1 \rangle$ onto (along) $\mathbf{a} = \langle 4, 2 \rangle$.

Stewart's notations $\text{comp}_{\mathbf{a}}\mathbf{b}$ and $\text{prog}_{\mathbf{a}}\mathbf{b}$ are not standard, and I won't use them. But I can't guarantee that they won't show up on common exams.

Intuitively, a projection of $\mathbf{b}$ onto $\mathbf{a}$ is "the piece of $\mathbf{b}$ that points in the direction of $\mathbf{a}$". It shouldn't depend on the length of $\mathbf{a}$, and in fact

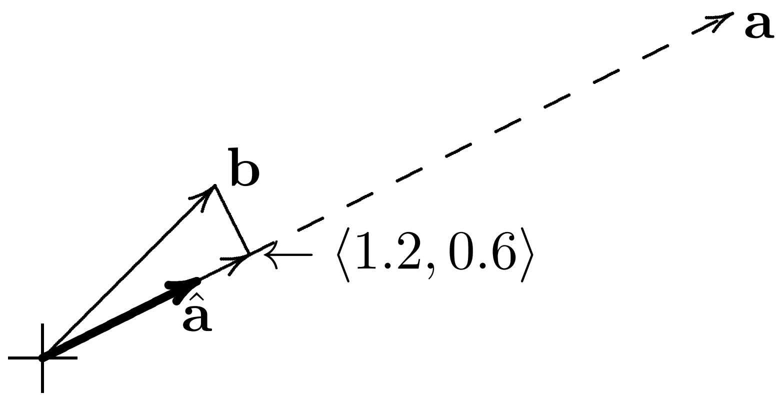
$$\hat{\mathbf{a}} \cdot \mathbf{b} = |\mathbf{b}| \cos\theta$$

is the line-segment length called the scalar projection. Then

$$(\hat{\mathbf{a}} \cdot \mathbf{b})\hat{\mathbf{a}}$$

is the corresponding vector projection. (Compare the formulas and drawings on p. 58. Also see my drawing on the next slide.)

$\mathbf{a} = \langle 4, 2\rangle$, so the relevant unit vector is $\hat{\mathbf{a}} = \frac{\langle 4,2\rangle}{\sqrt{16+4}} = \frac{1}{\sqrt{5}}\langle 2, 1\rangle$. And $\mathbf{b} = \langle 1, 1\rangle$, so the scalar projection is $\hat{\mathbf{a}} \cdot \mathbf{b} = \frac{3}{\sqrt{5}}$. The vector projection thus is $(\hat{\mathbf{a}} \cdot \mathbf{b})\hat{\mathbf{a}} = \frac{3}{5}\langle 2, 1\rangle$.



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