Lecture for Week 10 (Secs. 4.5–8)

Derivative Miscellany III

Growth and decay problems

I already discussed the theory of these problems two weeks ago, so let's just do an example.

Exercise 4.5.3

A culture starts with 500 bacteria, and after 3 hours there are 8000.

(a) Find the formula for the number after t hours.

(b) Find the number after 4 hours.

(c) When will the population reach 30,000?

The basic assumption is that the number of new bacteria is proportional to the number already there (parents). So

$$B(t) = B(0)e^{kt}$$

for some constant k. So according to the data,

$$8000 = 500e^{3k},$$

or $3k = \ln \frac{80}{5} = \ln 16$, or $k = \frac{1}{3} \ln 16$.

$$B(t) = 500e^{\frac{1}{3} t \ln 16} = 500(16)^{t/3}.$$

Then

$$B(4) = 500(16)^{4/3} = \text{(calculator output)}.$$

For the last part,

$$\begin{aligned} 30,000 = 500(16)^{t/3} \; &\Rightarrow \; \frac{t}{3} = \log_{16} \frac{300}{5} \\ &\Rightarrow \; t = 3 \log_{16} 60 \\ &\qquad = 3\, \frac{\ln 60}{\ln 16}\,. \qquad \blacksquare \end{aligned}$$

Inverse trigonometric functions

There are two aspects of inverse trig functions that need to be studied:

- the definitions (especially branch choices);
- their derivatives.

The most important inverse trig functions are $\sin^{-1}$ and $\tan^{-1}$.

Both of the problems we encountered for the square root function also appear for the inverse sine.

1. $\sin\theta$ is always between -1 and 1, so there is no way to define $\sin^{-1} z$ if $|z| > 1$ (unless we go into complex numbers, which we won't).

2. For $-1 \leq z \leq 1$ there is more than one θ with z as its sine. (In fact, there are infinitely many.) So, we have to choose a prin-

cipal value (or branch) of the inverse function. The standard choice is to pick θ so that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Please refer to the book for the graphs (p. 276 and p. 278 in this case). Recall that to get the graph of an inverse function, you can plot the original function on a transparent sheet and flip it over so that the horizontal and vertical axes are interchanged.

Thus $\sin(\sin^{-1} z) = z$ always, but $\sin^{-1}(\sin\theta) = \theta$ is false if θ is not in the principal interval.

Recall also that $\sin^{-1} z$ **does not mean** $(\sin z)^{-1}$ (that is, $1/\sin z$), although $\sin^2 z$ **does** mean $(\sin z)^2$. This notational inconsistency is unfortunate, but we're stuck with it. (Let's not even ask what $\sin^{-2} z$ means.) Another notation for the inverse is $\arcsin z$.

The inverse tangent is easier (see graphs p. 279), because it is defined for all z and all the branches look the same (have positive slope). But there are still infinitely many branches, and the standard choice is $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. (Why is it "$<$" here but "$\leq$" for the inverse sine?)

This $\tan^{-1} z$ is a very nice function. It increases smoothly between horizontal asymptotes at $\theta = -\frac{\pi}{2}$ and $\theta = +\frac{\pi}{2}$.

The usual technique for differentiating an implicit or inverse function yields the formulas

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}\,, \qquad \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}\,.$$

These are ordinary algebraic functions! (All trace of trig seems to have disappeared.) One reason inverse trig functions are important is that they help provide the antiderivatives of certain algebraic functions.

Exercise 4.6.51

Find the derivative of $g(x) = \sin^{-1}(3x+1)$ and state the domains of g and g'.

Exercise

Find an antiderivative of

$$f(x) = \frac{3}{\sqrt{4-4x^2}} - \frac{10}{x^2+1}.$$

$$g(x) = \sin^{-1}(3x+1).$$

$$g'(x) = \frac{3}{\sqrt{1-(3x+1)^2}}$$

(which could be simplified). For g to be defined we need $|3x+1| \leq 1$.

Case 1: $3x+1 \geq 0$. Then

$$3x+1 \leq 1 \;\Rightarrow\; x \leq 0,$$

$$3x+1 \geq 0 \;\Rightarrow\; x \geq -\tfrac{1}{3}\,.$$

Case 2: $3x + 1 < 0$. Then

$$-3x - 1 \leq 1 \;\Rightarrow\; x \geq -\tfrac{2}{3}\,,$$

$$3x + 1 < 0 \;\Rightarrow\; x < -\tfrac{1}{3}\,.$$

So the domain of g consists of the two intervals

$$-\tfrac{1}{3} \leq x \leq 0 \quad \text{and} \quad -\tfrac{2}{3} \leq x < -\tfrac{1}{3}\,,$$

which fit together to give

$$-\tfrac{2}{3} \leq x \leq 0.$$

For g' to be defined we **also** need $3x + 1 \neq 0$, hence the interval shrinks to $-\frac{2}{3} < x \leq 0$. (See the vertical tangents at the ends of the graph, Fig. 4 on p. 278.)

Alternative solution of the inequality:

$$|3x + 1| \leq 1 \iff \left| x + \tfrac{1}{3} \right| \leq \tfrac{1}{3}\,.$$

This clearly describes the numbers whose distance from $-\frac{1}{3}$ is at most $\frac{1}{3}$ — namely, the interval $\left[-\frac{2}{3}, 0\right]$. ■

$$F'(x) = \frac{3}{\sqrt{4-4x^2}} - \frac{10}{x^2+1}\,;$$

what is F?

$$F'(x) = \frac{3}{2}\frac{1}{\sqrt{1-x^2}} - \frac{10}{x^2+1}\,,$$

so the obvious choice is

$$F(x) = \frac{3}{2}\sin^{-1} x - 10\tan^{-1} x.$$

Soon we will reach the proof that the only other antiderivatives are equal to this one plus a constant.

(What if the two numbers inside the square root were not the same? Look forward to the excitement of Chapter 8 in Math. 152!) ■

Hyperbolic functions

This topic is not in the syllabus for Math. 151 at TAMU. To see why it should be, read my paper in *College Math. Journal* **36** (2005) 381–387. It also explains why I don't talk about cot, csc, $\sec^{-1}$, etc.

Indeterminate forms (l'Hospital's rule)

(The name is pronounced “Loap-it-ALL” (more or less) and sometimes spelled “l'Hôpital”.)

In my opinion, the two most important things to learn about l'Hospital's rule are

- when **not** to use it
- what it teaches us about limits of exp and ln at infinity.

Suppose we want to calculate the limit of $\frac{f(x)}{g(x)}$ as $x \to a$ (a may be ∞), and suppose that **both** $f(x)$ and $g(x)$ approach 0 in that limit, or both approach ∞. L'Hospital's rule states that that limit is the same as the limit of $\frac{f'(x)}{g'(x)}$ (which may be easier to calculate).

Please don't confuse this formula with the "limit law" for a quotient, or with the formula for the derivative of a quotient. They are three different things!

Here is an example of the correct use of the rule:

$$\lim_{x\to 0} \frac{\sin(5x)}{7x} = \text{“}\frac{0}{0}\text{”}$$
$$= \lim_{x\to 0} \frac{5\cos(5x)}{7} = \frac{5}{7}\,.$$

However, you didn't really need the rule to do this problem, did you? You already know that $\sin(5x) \approx 5x$ when $x \approx 0$ (or can appeal to $\lim\limits_{u\to 0} \frac{\sin u}{u} = 1$).

After studying Taylor series (Chapter 10) you will know many other situations where the behavior of the functions f and g near a is obvious, so l'Hospital is unnecessary. Many students overuse l'Hospital's rule, relying on it as a "black box" when they would learn much more (and solve the problems equally fast) by just taking a close look at, and comparing, the behavior of the numerator and denominator as $x \to a$.

Here is an example where using the rule is absolutely wrong: We know that $\lim_{x\to 0^+} \frac{\cos x}{x} = +\infty$, because the numerator approaches 1 while the denominator approaches 0. If you incorrectly applied l'Hospital's rule, you would get $\lim_{x\to 0^+} \frac{-\sin x}{1} = 0$. This fraction *did not satisfy the hypotheses* of l'Hospital's rule, because the limit of the numerator, $\cos x$, is not 0.

Finally, consider $\lim_{x \to 0} (x \ln x)$. This is an indeterminate form of the type "$0 \times \infty$". To apply l'Hospital's rule we must rewrite it as a quotient.

First try: $\displaystyle\lim_{x \to 0^+} \frac{x}{(\ln x)^{-1}}$ is an indeterminate form of type $\frac{0}{0}$. The rule gives

$$\lim_{x \to 0^+} \frac{1}{-\frac{1/x}{(\ln x)^2}} = \lim_{x \to 0^+} [-x(\ln x)^2].$$

This has made the problem worse!

Second try: $\displaystyle\lim_{x\to 0^+} \frac{\ln x}{1/x}$ is an indeterminate form of type $\frac{\infty}{\infty}$. The rule gives

$$\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

The second method works; the first one doesn't.

In summary,

1. L'Hospital's rule can give the *wrong answer* if the conditions for its validity are not satisfied.

2. Sometimes it does not lead to an answer, it just makes the problem more complicated.

3. Sometimes it works but there is a better way of solving the problem.

4. But sometimes it works and is very useful!

When *should* you use l'Hospital's rule? By far the most important situation is when the numerator or denominator does not have an obvious power-like behavior as x approaches a. This is the case for the logarithm function as x approaches either 0 or infinity, and for the exponential function as x approaches either positive or negative infinity. (That is why the section on l'Hospital's rule is in this chapter!)

In homework you'll take some limits of ratios of exponentials, logarithms, and ordinary powers. After awhile the results of such calculations become very predictable. They can be summarized in a list of general conclusions:

1. As $x \to +\infty$, e^x increases faster than any power, x^n.

2. As $x \to +\infty$, e^{-x} decreases faster than any negative power, x^{-n}. (Equivalently: As $x \to$

$-\infty$, e^{+x} decreases faster than any negative power.)

3. As $x \to +\infty$, $\ln x$, although it goes to infinity, increases more slowly than any positive power, x^a (even a fractional power such as $a = \frac{1}{200}$).

4. As $x \to 0^+$, $-\ln x$ goes to infinity, but more slowly than any negative power, x^{-a} (even a fractional one).

There are some other famous indeterminate forms, 0^0, ∞^0, and 0^∞. Note that

$$f(x)^{g(x)} = e^{g(x) \ln f(x)}.$$

Therefore, these 3 cases arise when $g \ln f$ is an indeterminate form of the $0 \times \infty$ type (which can happen in 3 different ways, depending on whether the log approaches 0, $+\infty$, or $-\infty$). So, solve the $0 \times \infty$ problem and exponentiate the result.

Very often, but **not always**, the answer to a 0^0 problem will be 1. For more information on this topic, see the Web page "`lhop.htm`" linked to our home page. (You can skip over the first part, which is an older version of everything I have just told you about l'Hospital's rule.)