Math. 309 (Fulling) 11 Dec. 2018

Final Examination – Solutions

Name: ____________________

Calculators may be used for simple arithmetic operations only!

1. *(20 pts.)*

(a) What is the dimension of the space of 3×3 matrices?

9 (There are 9 independent matrix elements.)

(b) What is the dimension of the space of 3×3 **antisymmetric** matrices?

3 (A typical member is $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$.)

(c) What is the dimension of the space of 3×3 **symmetric** matrices?

6 (A typical member is $\begin{pmatrix} d & a & b \\ a & e & c \\ b & c & f \end{pmatrix}$.)

(d) Show that every matrix is the sum of a symmetric and an antisymmetric matrix.

Write $M = \frac{1}{2}(M + M^{\mathrm{t}}) + \frac{1}{2}(M - M^{\mathrm{t}})$ and note that the first term is symmetric and the second term is antisymmetric.

Remark: The only matrix that is both symmetric and antisymmetric is 0. Therefore, the full matrix space is the *direct sum* of its symmetric and antisymmetric subspaces, and it is no surprise that the dimensions add: $9 = 3 + 6$.

2. *(25 pts.)* Consider $f(x) = x$ defined on the domain $0 < x < 1$.

(a) Find the Fourier **cosine** series of f. (Yes, you should be able to evaluate the integrals.)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

where

$$a_n = 2\int_0^1 f(x)\cos(n\pi x)\,dx.$$

Thus

$$a_0 = 2\int_0^1 x\,dx = 1,$$

and for larger n (with $u = x$, $v = \sin(n\pi x)/(n\pi)$)

$$a_n = 2\int_0^1 x\cos(n\pi x)\,dx = \frac{2x}{n\pi}\sin(n\pi x)\Big|_0^1 - \frac{2}{n\pi}\int_0^1 \sin(n\pi x)\,dx$$

$$= 0 - \frac{2}{n\pi}\left(-\frac{1}{n\pi}\right)\cos(n\pi x)\Big|_0^1 = \frac{2}{n^2\pi^2}[\cos(n\pi) - 1].$$

If n is even (and >0), this is 0. If n is odd, write $n=2k-1$ and get

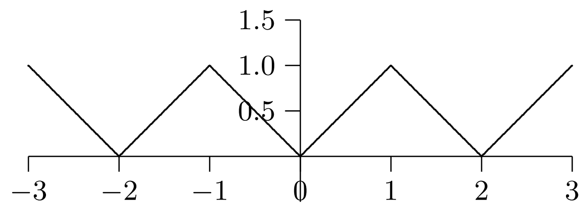
$$a_{2k-1} = -\frac{4}{(2k-1)^2\pi^2}\,.$$

Thus

$$x = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2} \cos\bigl((2k-1)\pi x\bigr).$$

(b) To what does the series converge, at points x outside the original domain? Explain in words and by sketching the graph of the function. (Don't try to give formulas.)

The limit function is the even periodic extension of f, which is a triangle wave (peaks when x is an odd integer, troughs when n is an even integer).



3. *(25 pts.)*

(a) Find all solutions of $\begin{cases} x \quad\;\; + z = 0, \\ x + y - z = b, \\ 3x + y + z = c, \end{cases}$ where b and c are arbitrary constants.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & b \\ 3 & 1 & 1 & c \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & b \\ 0 & 1 & -2 & c \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & b \\ 0 & 0 & 0 & c-b \end{pmatrix}.$$

Clearly there is *no solution* if $b \neq c$. If $b = c$, then

$$z \text{ is arbitrary}, \quad y = b + 2z, \quad x = -z.$$

(b) Discuss the kernel, rank, and range of the linear function with matrix $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix}$.

The kernel is the case $b = c = 0$ of (a):

$$\text{kernel} = \text{span}\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

Therefore, the rank (dimension of the range) equals 2 (by the rank-nullity theorem). There are several ways to determine the range:

Method 1 (traditional): Column-reduce the matrix (i.e., row-reduce its transpose), getting

$$\text{range} = \operatorname{span}\left\{\begin{pmatrix}1\\0\\2\end{pmatrix},\quad\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}.$$

Method 2: From (a), it is obvious that multiples of $\begin{pmatrix}0\\1\\1\end{pmatrix}$ are in the range. Since the rank is 2, we know that we need another linearly independent vector. Any of the columns of the original matrix (except the middle one), or any linear combination of them, will do. The result is a basis equivalent to the one found by the first method.
Method 3: Redo (a) with the first equation replaced by $x + z = a$. Conclusion: $c = b + 2a$. Then the basis vectors found by the first method fall out from the choices $a = 1$, $b = 0$ and $a = 0$, $b = 1$.

4. *(25 pts.)* The temperature in a ring of metal is determined by the heat equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial \theta^2}$ and initial data $u(0, \theta) = f(\theta)$. Here θ is a periodic (angular) coordinate, so $u(t, \theta + 2\pi) = u(t, \theta)$ and $f(\theta + 2\pi) = f(\theta)$.

   Solve the problem by separation of variables. (Of course, you won't be able to evaluate the integrals for the Fourier coefficients, but you can write down the formulas.)

First look for separated solutions: $u = T(t)Y(\theta)$, $T'Y = TY''$,

$$\frac{T'}{T} = \frac{Y''}{Y} = -\lambda^2.$$

The spatial (angular) solutions must be $Y = A\cos(\lambda\theta) + B\sin(\lambda\theta)$, where λ must be real to have any hope of periodicity, $\lambda < 0$ can be omitted because they duplicate the positive ones, and periodicity with period 2π dictates that λ must be an integer. This includes $\lambda = 0$, but only the cosine (constant) term is allowed in that case. (The analog of the sine term for $\lambda = 0$ is Bx, but that is not periodic.) The temporal solution then is $T = e^{-n^2 t}$.

The general solution is a sum of such terms:

$$u(t, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-n^2 t}[a_n \cos(nx) + b_n \sin(nx)].$$

The coefficients must be found from the initial condition. The form of that Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]$$

with

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta\,, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta\,.$$

(The integrals can also be written over the interval from $-\pi$ to π; because the integrands are periodic with period 2π, this doesn't change the values of the integrals. Under test conditions you get almost full credit even if you don't get the numerical factors exactly right.) These formulas can be found on p. 258 of Leon, and also on p. 11 of Vogel with $L = \pi$.

5. *(24 pts.)* Define an inner product on (real-valued) functions by $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$. Find the first 3 of the normalized orthogonal polynomials corresponding to this inner product.

These are called *Legendre polynomials*. Leon discusses them on p. 278 but calculates them by a method we did not discuss and lists them in an unnormalized form. The Gram–Schmidt calculation and the orthonormalization are given on p. 288 of my book *Linearity* (which is still visible to you for another month, at least).

6. *(30 pts.)*

(a) Find all the eigenvalues and eigenvectors of $B = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

Eigenvalues: $0 = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5$. Thus

$$\lambda = \tfrac{1}{2}(4 \pm \sqrt{16+20}) = \tfrac{1}{2}(4 \pm 6) = \begin{cases} 5 \\ -1 \end{cases}.$$

Eigenvectors with $\lambda = 5$:

$$\begin{pmatrix} -4 & 2 & 0 \\ 4 & -2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x = \frac{1}{2}y.$$

The eigenvectors are the multiples of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Eigenvectors with $\lambda = -1$:

$$\begin{pmatrix} 2 & 2 & 0 \\ 4 & 4 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x = -y.$$

The eigenvectors are the multiples of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(b) Find a diagonal matrix D and a matrix Q such that $B = QDQ^{-1}$ or $D = QBQ^{-1}$. **State WHICH of these two formulas applies!**

Let

$$D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}.$$

Since Q has the eigenvectors as columns, it maps coordinates with respect to the eigenbasis into coordinates with respect to the original or natural basis. Therefore, the correct equation is $B = QDQ^{-1}$ (which you can verify by calculating Q^{-1} and doing the matrix multiplications). *Note:* You can write the eigenvalues in the other order in D, but then your Q must change accordingly. Also, if you normalized your eigenvectors differently, then you got a different Q; in the end the differences are cancelled by the resulting differences in Q^{-1}.

Remark: Kudos to Ben Sarawichitr for observing that $D = QBQ^{-1}$ is *also* true in this problem. The formula that is always true (if Q and D are defined as I did here) is $D = Q^{-1}BQ$, but in this problem $Q^{-1} = -\frac{1}{3}Q$ by accident.

(c) Use D and Q to solve the ODE system $\frac{dx}{dt} = x + 2y, \quad \frac{dy}{dt} = 4x + 3y$ with arbitrary initial data $x(0)$, $y(0)$.

Method 1:

$$e^{tD} = \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{-t} \end{pmatrix};$$

$$\begin{aligned} e^{tB} = Qe^{tD}Q^{-1} &= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{-t} \end{pmatrix} \left(\frac{-1}{3}\right) \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} e^{5t} & e^{-t} \\ 2e^{5t} & -e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} e^{5t} + 2e^{-t} & e^{5t} - e^{-t} \\ 2e^{5t} - 2e^{-t} & 2e^{5t} + e^{-t} \end{pmatrix}. \end{aligned}$$

Then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tM} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

Method 2: The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} = \begin{pmatrix} e^{5t} & e^{-t} \\ 2e^{5t} & -e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = Qe^{tD} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

To determine the c_j we look at the initial data:

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = Qe^{0} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = Q^{-1} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

7. *(Essay question – 20 pts.)* Tell me some things you know about Bessel functions.

This one is up to you.

8. *(16 pts.)*

(a) Define “subspace”.

A subspace is a nonempty subset of a vector space that is *closed* under addition and scalar multiplication. In other words, if $\vec{v}_1$ and $\vec{v}_2$ are in the set, then so are all the linear combinations $r\vec{v}_1 + \vec{v}_2$ (for any scalar r).

(b) Prove that the kernel (also known as “nullspace”) of a linear function is a sub**space** (not just a sub**set**) of the function’s domain.

The kernel consists of all vectors $\vec{v}$ such that $L(\vec{v}) = 0$. If $\vec{v}_1$ and $\vec{v}_2$ are in the kernel, then since L is linear,

$$L() = rL(\vec{v}_1) + L(\vec{v}_2) = 0 + 0 = 0.$$

Thus $r\vec{v}_1 + \vec{v}_2$ is in the kernel. (The kernel is nonempty because it contains the 0 vector.)

9. *(15 pts.)* Using some well-known tricks to simplify the task, calculate the determinant
$$\begin{vmatrix} 0 & -1 & 3 & 1 \\ 0 & 0 & 2 & 0 \\ 25 & 50 & 75 & 100 \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

There are many routes, but I think this is the best:

$$(-2)\begin{vmatrix} 0 & -1 & 1 \\ 25 & 50 & 100 \\ 1 & 1 & 1 \end{vmatrix} = (-50)\begin{vmatrix} 0 & -1 & 1 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \end{vmatrix} = (-50)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= (-50)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 3 \end{vmatrix} = (+50)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{vmatrix} = (50)(1)(1)(4) = 200.$$