Math. 309 (Fulling) 4 October 2018

Test A – Solutions

Name: ____

Calculators may be used for simple arithmetic operations only!

1. *(18 pts.)* Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$.

Calculate each of these, or declare it undefined:

(a) $A + B$

undefined

(b) $A + C$

$\begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$

(c) AB

$\begin{pmatrix} 7 & 2 & -2 \\ 21 & 6 & -6 \end{pmatrix}$

(d) BC

undefined

(e) A^{-1}

undefined

(f) C^{-1}

$\frac{1}{2}\begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}$

2. *(16 pts.)* Find all solutions of each of these systems.

(a) $\begin{cases} x + 2y = 2, \\ 2x + 4y = 2. \end{cases}$

$$\left(\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & 4 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & -2 \end{array}\right)$$

Hence there are no solutions.

(b) $\begin{cases} x + 2y - 3z = 0, \\ x - \ \ y + 2z = 3. \end{cases}$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 1 & -1 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 1 & 2 & -3 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -5 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 2 \\ 0 & 1 & -\frac{5}{3} & -1 \end{array}\right)$$

That is,

$$x = 2 - \frac{1}{3}z, \qquad y = -1 + \frac{5}{3}z, \qquad z \text{ arbitrary.}$$

3. *(18 pts.)* Determine whether each set is linearly independent. If it is not, find an independent set with the same span.

(a) $$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Independent.

(b) $$\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 5 \\ 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus these vectors are dependent. The nonzero rows of the reduced matrix transpose to two independent column vectors (which happen, by accident, to be the same as the first and third of the original vectors). There are other correct two-vector sets.

(c) $\{t^2 - 1, \quad t + 1, \quad t^2 + 1\}$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus the original polynomials are independent. (But from the reduced matrix we can read off a simpler basis: $\{t^2, \ t, \ 1\}$.)

4. *(10 pts.)* **Do ONE of these [(A) or (B)].** (Up to 5 points extra credit for doing the other one.)

(A) Producing a refrigerator requires 0.2 ton of steel and 0.2 ton of plastic. Producing an airplane requires 6 tons of steel and 2 tons of plastic. Producing a ton of steel consumes 4 tons of coal and 10 barrels of water. Producing a ton of plastic consumes 2 tons of coal and 30 barrels of water. Organize these facts into matrices, and find the matrix that tells you how much coal (c) and water (w) is needed to make r refrigerators and a airplanes.

Let s and p be the quantities of steel and plastic, and let

$$\begin{pmatrix} s \\ p \end{pmatrix} = B \begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 0.2 & 6 \\ 0.2 & 2 \end{pmatrix} \begin{pmatrix} r \\ a \end{pmatrix}, \qquad \begin{pmatrix} c \\ w \end{pmatrix} = A \begin{pmatrix} s \\ p \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 10 & 30 \end{pmatrix} \begin{pmatrix} s \\ p \end{pmatrix}.$$

Then $\begin{pmatrix} c \\ w \end{pmatrix} = AB \begin{pmatrix} r \\ a \end{pmatrix}$, where

$$AB = \begin{pmatrix} 4 & 2 \\ 10 & 30 \end{pmatrix} \begin{pmatrix} 0.2 & 6 \\ 0.2 & 2 \end{pmatrix} = \begin{pmatrix} 1.2 & 28 \\ 8 & 120 \end{pmatrix}.$$

(B) On the island of Lavonia the farmers keep $\frac{1}{2}$ of their produce and give $\frac{1}{4}$ to the blacksmiths and $\frac{1}{4}$ to the musicians. The blacksmiths give $\frac{2}{3}$ of their production to the farmers and keep the rest for themselves. The musicians can be heard by everyone, but because there are so few of them and so many farmers, the effective distribution of their benefits is $\frac{1}{2}$ to the farmers, $\frac{1}{3}$ to the blacksmiths, and $\frac{1}{6}$ to the musicians themselves. Write down the (closed) Leontief matrix, L, representing this economy, and write down (don't solve) the equation system determining the equilibrium production levels.

With the rows and columns labeled in the order shown, the matrix is

$$L = \begin{matrix} \\ \\ F \leftarrow \\ B \leftarrow \\ M \leftarrow \end{matrix} \begin{matrix} F & B & M \\ \downarrow & \downarrow & \downarrow \\ \end{matrix} \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{1}{6} \end{pmatrix}.$$

Equilibrium occurs if $L\vec{X} = \vec{X}$ — that is,

$$\begin{aligned} \tfrac{1}{2}F + \tfrac{2}{3}B + \tfrac{1}{2}M &= F, \\ \tfrac{1}{4}F + \tfrac{1}{3}B + \tfrac{1}{3}M &= B, \\ \tfrac{1}{4}F \qquad\quad + \tfrac{1}{6}M &= M. \end{aligned}$$

In other words, we need the solutions of $(L - I)\vec{X} = 0$, which we could find by row-reducing

$$\begin{pmatrix} -\frac{1}{2} & \frac{2}{3} & \frac{1}{2} \\ \frac{1}{4} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & 0 & -\frac{5}{6} \end{pmatrix}.$$

5. *(8 pts.)* Calculate the determinant $\begin{vmatrix} 3 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix}$.

Let's expand in minors of the top row:

$$\text{Det} = 3\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} -2 & 1 \\ 2 & -1 \end{vmatrix} = 6 + 4 + 0 = 10.$$

6. *(18 pts.)*

(a) Let $\mathcal{V}$ be the set of all 2×2 matrices (with real entries). Show that $\mathcal{V}$ is a vector space, given the usual definitions of addition and scalar multiplication of matrices.

Quick answer: This space is obviously the same as $\mathbf{R}^4$ in disguise. It doesn't matter whether you write the 4 entries in a square, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or in a column, $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$. (Note: addition and scalar multiplication are entry-by-entry in either case, so they are really the same operation in either notation.)

Detailed answer: Verify the 8 axioms (Leon p. 115). Verify them for *matrices*, not for the vectors in $\mathbf{R}^2$. (You should also remark that the set is closed under the two operations.)

(b) What is the dimension of this vector space?

4 (not 2!)

7. *(12 pts.)* Find the inverse (if it exists) of the matrix $M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ -2 & 4 & -1 \end{pmatrix}$.

Indicate what row operations you're performing.

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ -2 & 4 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{(1)\leftrightarrow(2)} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ -2 & 4 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{(3)\to(3)+2(1)} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 8 & 1 & 0 & 2 & 1 \end{array}\right) \xrightarrow[(2)\leftrightarrow(3)]{(3)\to\frac{1}{8}(3)} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{8} & 0 & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[(2)\to(2)-\frac{1}{8}(3)]{(1)\to(1)-2(2)} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{4} & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{(1)\to(1)-\frac{3}{4}(3)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

Thus

$$M^{-1} = \begin{pmatrix} -\frac{3}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 1 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \frac{1}{8}\begin{pmatrix} -6 & 4 & -2 \\ -1 & 2 & 1 \\ 8 & 0 & 0 \end{pmatrix}.$$