Math. 309 (Fulling) 8 November 2018

Test B – Solutions

Name: ______________________________

Calculators may be used for simple arithmetic operations only!

1. *(9 pts.)* Which of these are **subspaces** of $\mathbf{R}^3$?

(a) The solutions of $x - 2y + z = 1$

NO. Not closed under scalar multiplication (for example) because of the nonhomogeneous term.

(b) The solutions of $2x - 3y = 0$

YES.

(c) The solutions of $x^2 + y + z = 0$

NO. Not closed under scalar multiplication or addition because of the nonlinear term, x^2 .

2. *(10 pts.)* On $\mathbf{R}^3$ let's use the notation $\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, etc. Define

$$\langle \vec{v}_1, \vec{v}_2 \rangle = {x_1}^2 {x_2}^2 + {y_1}^2 {y_2}^2 + {z_1}^2 {z_2}^2 .$$

Does this formula define an inner product on $\mathbf{R}^3$? Explain.

NO. Again the squares spoil it. An inner product must be bilinear, symmetric, and positive definite. This one is not bilinear (linear in each $\vec{v}_j$ by itself when the other vector is fixed). It does satisfy the other two properties.

3. *(16 pts.)* The linear function $F\colon \mathbf{R}^2 \to \mathbf{R}^2$ is represented by $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ with respect to the natural basis.

(a) Find the matrix representing F if the basis in the domain is changed to

$$\left\{ \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$

(the basis for the codomain remaining unchanged).

Let's put the new basis vectors together into the matrix $B = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$. B acts on coordinates of a vector with respect to the new basis to give the coordinates of that vector with respect to the natural basis.

To get the desired matrix for the linear function we need to preprocess the new coordinates of the domain vector back into the old basis so that A can act upon them. This is exactly what B does, so the answer is

$$AB = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 7 & 4 \end{pmatrix} .$$

(b) Find the matrix representing F if that basis is used for the codomain (the basis for the domain remaining unchanged).

This time we need to postprocess the output from A the translate it from the natural basis to the new basis. That means we multiply A on the left by B^{-1}.

$$B^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}.$$

$$B^{-1}A = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ 3 & 6 \end{pmatrix}.$$

Check: From both of these results we compute

$$A' \equiv B^{-1}AB = \frac{1}{5}\begin{pmatrix} 11 & 2 \\ 12 & 9 \end{pmatrix}.$$

Getting the same answer both ways is reassuring, but we can go further: A' is the matrix of the linear function with respect to the new basis at both ends. Under such a similarity transformation, the trace and determinant must be unchanged, and they are:

$$\text{trace} = 4, \quad \det = 3.$$

4. *(30 pts.)* Let $\mathcal{V}$ be the vector space of quadratic polynomials (called $\mathcal{P}_2$ by Fulling but $\mathcal{P}_3$ by Leon). Let $L\colon \mathcal{V} \to \mathcal{V}$ be the differential operator $(Lp)(t) \equiv p''(t) + tp'(t)$.

(a) Find the matrix that represents L with respect to the standard basis $\{t^2, t, 1\}$ for $\mathcal{V}$.

$$L(t^2) = 2t^2 + 2, \quad L(t) = t, \quad L(1) = 0.$$

Put these results into the matrix according to the kth-column rule:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

(b) Find the kernel of L. Is L injective?

Row-reduce the matrix: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It follows that, in the notation $p(t) = at^2 + bt + c$, the elements of the kernel are the polynomials with $a = 0$, $b = 0$, c arbitrary; in other words, the constant functions.

The function is not injective, since it has a nontrivial kernel.

(c) Find the range of L. Is L surjective?

Transpose the matrix and row-reduce to get a simple basis for the range:

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The span therefore is all the polynomials of the form $p(t) = a(t^2 + 1) + bt$.

The function is not surjective, since its range is not the entire space $\mathcal{V}$.

(d) Use "superposition" to find **all** polynomial solutions in $\mathcal{V}$ of the differential equation $p''(t) + tp'(t) = t^2$.

Everybody got full credit on this part, because as written it does not have a simple solution as intended. Let's change the question so it makes sense:

Corrected problem: Use "superposition" to find **all** polynomial solutions in $\mathcal{V}$ of the differential equation $p''(t) + tp'(t) = t^2 + 1$.

From the calculations in part (a) we see that one solution is $p_0(t) = \frac{1}{2}t^2$. The most general solution is $p(t) = p_0(t) + p_{\rm h}(t)$, where $p_{\rm h}$ is the most general element of the kernel (the general solution of the homogeneous equation). In other words,

$$p(t) = \frac{t^2}{2} + c \quad \text{for any constant } c\,.$$

Comment: This is a second-order equation, so shouldn't we have a two-parameter family of solutions? Well, yes, but the other solutions are not members of $\mathcal{V}$; in fact, they are not polynomials at all. The missing solutions are $Ce^{-t^2/2}$ with arbitrary constant C, as you should be able to discover by Math. 308 methods.

5. *(20 pts.)* Let $M = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(a) Find *all* eigenvalues and eigenvectors of M.

Eigenvalues: $0 = \begin{vmatrix} 3-\lambda & 1 & 0 \\ 1 & 3-\lambda & \\ 0 & 0 & 2-\lambda \end{vmatrix}$

$$= (2-\lambda)[(3-\lambda)^2 - 1] = (2-\lambda)(\lambda^2 - 6\lambda + 8] = (\lambda-2)^2(\lambda-4).$$

So the eigenvalues are 2 and 4.

Eigenvectors for $\lambda = 2$*:* Insert this value of λ into the matrix inside the determinant, and reduce:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus z is arbitrary and $x + y = 0$. The most general eigenvector is of the form $\begin{pmatrix} x \\ -x \\ z \end{pmatrix}$.

An orthonormal basis for this eigenspace is

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Eigenvectors for $\lambda = 4$*:* $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Thus $z = 0$ and $x = y$. The general vector is $\begin{pmatrix} x \\ x \\ 0 \end{pmatrix}$.

A normalized eigenvector is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$.

(b) Find an *orthogonal* matrix U that diagonalizes M (that is, $M = UDU^{-1}$).

Let's write the diagonal matrix as $D = \begin{pmatrix}4&0&0\\0&2&0\\0&0&2\end{pmatrix}$. Put the eigenvectors in that same order into the columns of the basis-change matrix:

$$O = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1&0\\1&-1&0\\0&0&\sqrt{2}\end{pmatrix}. \qquad O^{-1} = O^{\mathrm{t}} = O.$$

(The fact that O is symmetric is an accident. It would not have happened if I had written the eigenvalues and eigenvectors in a different order, or chosen the opposite sign for the eigenvector containing a negative number.) O maps coordinates from the eigenbasis to the natural basis, which is what we want U to do in the problem as written. (The coordinate transformation is running here in the opposite direction compared to Question 3.) So $U = O$, even in the cases where $O \neq O^{\mathrm{t}}$.

6. *(15 pts.)* Consider the inner product $\langle p, q\rangle \equiv \int_{-\infty}^{\infty} p(t)\,q(t)\,e^{-t^2}\,dt$ on the vector space of all polynomials. When the Gram–Schmidt procedure is applied to the standard basis $\{1, t, t^2, \ldots\} \equiv \{v_0, v_1, v_2, \ldots\}$, the elements of the resulting orthonormal basis are called *Hermite polynomials*. Find the first 3 Hermite polynomials. FREE INFORMATION:

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}, \quad \int_{-\infty}^{\infty} t^2\,e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}, \quad \int_{-\infty}^{\infty} t^4\,e^{-t^2}\,dt = \frac{3\sqrt{\pi}}{4}.$$

Note first that whenever m is *odd*,

$$\int_{-\infty}^{\infty} t^m\,e^{-t^2}\,dt = 0$$

by symmetry (the contribution from negative t exactly cancels that from positive t), which is why those integrals did not need to be tabulated in the hint. The square of the norm of the first vector is

$$\langle 1, 1\rangle = \int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi},$$

so

$$\hat{u}_0 = \pi^{-1/4}.$$

Next,

$$\langle 1, t\rangle = \int_{-\infty}^{\infty} t\,e^{-t^2}\,dt = 0,$$

so the second vector is already orthogonal to the first; all we need to do is normalize it:

$$\langle t, t\rangle = \int_{-\infty}^{\infty} t^2\,e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2},$$

so

$$\hat{u}_1 = \sqrt{2}\,\pi^{-1/4}\,t.$$

At the third step we have $\langle \hat{u}_1, t^2 \rangle = 0$ but

$$\langle \hat{u}_0, t^2 \rangle = \pi^{-1/4} \cdot \frac{\sqrt{\pi}}{2}\,,$$

so the part of t^2 parallel to $\hat{u}_0$ is $\frac{1}{2}$. Thus the perpendicular part is

$$t_\perp^2 = t^2 - \tfrac{1}{2}\,,$$

so that

$$\begin{aligned}\langle t_\perp^2, t_\perp^2 \rangle &= \int_{-\infty}^{\infty} (t^4 - t^2 + \tfrac{1}{4})\, e^{-t^2}\, dt \\ &= \frac{3\sqrt{\pi}}{4} - \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{4} = \frac{\sqrt{\pi}}{2}\,.\end{aligned}$$

Therefore,

$$\hat{u}_2 = \sqrt{2}\,\pi^{-1/4}\left(t^2 - \frac{1}{2}\right).$$