Math. 309 (Fulling) 3 Dec. 2020

Final Examination – Solutions

Upload your answers, in order, as a single document.

Calculators may be used for simple arithmetic operations only!

1. *(Multiple choice – each 5 pts.)*

(a) An example of a matrix of rank 1 is

(A) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(B) $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

(D) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

C

(b) If M is 3×3 matrix, then the determinant of $5M$ is

(A) $\det M$
(B) $5 \det M$
(C) $5^3 \det M$
(D) $5^9 \det M$

C

(c) The linearly independent eigenvectors of a matrix

(A) are always orthogonal.
(B) are always orthogonal if the matrix is symmetric.
(C) are orthogonal if the eigenvalues are distinct.
(D) are always orthogonal if the matrix is symmetric and the eigenvalues are distinct, and can be carefully chosen to be orthogonal if the matrix is symmetric and two eigenvalues are equal.

D

(d) If y_1 and y_2 are solutions of the same differential equation, then $y_1 + y_2$ is also a solution of that same equation, provided that the equation is

(A) linear.
(B) linear homogeneous.
(C) linear nonhomogeneous.
(D) nonlinear.

B

(e) Which of these is a subspace of the vector space $\mathbf{R}^3$?

(A) The space of solutions of $2x - 3y + z = 0$
(B) The space of solutions of $x^2 - 3y + z = 0$
(C) The space of solutions of $2x - 3 + z = 0$
(D) All of these

A

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2. *(20 pts.)*

(a) Define "subspace".

A subspace is a nonempty subset of a vector space that is closed under addition and scalar multiplication (equivalently, closed under formation of linear combinations of vectors).

(b) Define "range".

The range of a function is the set of elements (in the codomain) that occur as values of the function (as $f(x)$ for some x in the domain).

(c) Prove that the range of a linear function is a sub**space** (not just a sub**set**) of the function's codomain.

Let $\mathbf{u}$ and $\mathbf{v}$ be elements in the range. Then, by definition, $\mathbf{u} = f(\mathbf{x})$ and $\mathbf{v} = f(\mathbf{y})$ for some $\mathbf{x}$ and $\mathbf{y}$ in the domain. If r is any scalar, then

$$r\mathbf{u} + \mathbf{v} = rf(\mathbf{x}) + f(\mathbf{y}) = f(r\mathbf{x} + \mathbf{y})$$

since f is linear. Thus $r\mathbf{u} + \mathbf{v}$ is in the range.

3. *(Variable Separation "Jeopardy" – 20 pts.)* For each of these answers (labeled by lower-case letters), select from the list below (labeled by capital letters) what the question (PDE problem) was. *Don't try to solve the problems from scratch. Use general knowledge, your memory of Vogel's examples, and some attention to* **boundary conditions**.
Some coordinates have been renamed to avoid giving the game away; for example, "t" is not always a time variable here.

(a) $\displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\kappa n^2\pi^2 t/L^2}$

A

(b) $\displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi t}{L}\right)$

D

(c) $\displaystyle\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\kappa n^2\pi^2 t/L^2}$

B

(d) $\sum_{n=1}^{\infty} A_n J_0\left(\lambda_m x/L\right) e^{-\kappa \lambda_m{}^2 t/L^2}$

C

(A) Heat equation in a bar of length L, ends at zero temperature.
(B) Heat equation in a bar of length L, ends insulated (no heat flow).
(C) Heat equation in a disc, zero temperature on the circular boundary, initial temperature distribution independent of angle.
(D) Laplace's equation in a square of side length L, solution forced to be zero on three of the sides.

4. *(10 pts.)* Referring back to part (d) of the previous question, explain what J_0 and λ_m are.

J_0 is a Bessel function of order 0 (solution of $x^2y'' + xy' + x^2y = 0$), and λ_m is the m th positive root of that function (the m th place, counting away from the origin, where $J(\lambda) = 0$).

5. *(30 pts.)*

(a) Find the eigenvalues of $M = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$\det(M - \lambda) = \begin{vmatrix} 3-\lambda & 2 & 0 \\ 2 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = -(\lambda - 1)(\lambda^2 - 6\lambda + 5) = -(\lambda - 1)^2(\lambda - 5).$$

So the eigenvalues are 1 and 5.

(b) Find an orthonormal basis of eigenvectors of M.

Substitute the eigenvalues into the matrix formula $M - \lambda$, and row-reduce to find the kernel vectors:

$\lambda = 5:$
$$\begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $z = 0$ and $x = y$. A normalized vector is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

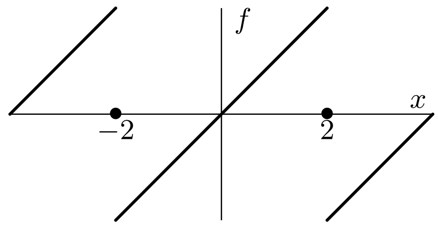
$\lambda = 1:$
$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus z is arbitrary and $x + y = 0$. Two orthonormal such vectors are

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

6. *(30 pts.)* Consider the Fourier sine series of $f(x) = x$ on the interval $0 < x < 2$.

(a) Without doing any calculations, sketch the graph of the function to which the series will converge, over the interval $-4 < x < 4$.



Why do we know immediately that the graph looks like this?
1. The series must converge to the given $f(x)$ on the given interval.
2. The limit must be an odd function of x.
3. The limit is periodic with period $2L$, where L is the length of the given interval.
4. At the jumps the series converges to the point in the middle of the gap (which will always be $y = 0$ for a pure sine series).

(b) Write down the series and the formula for the coefficients in it.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right),$$

$$b_n = \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx. \qquad \left(\frac{2}{L} = 1 \quad \text{because } L = 2.\right)$$

(c) Calculate the coefficients.

Integrate by parts:

$$b_n = -\frac{2}{n\pi}\int_0^2 x\, d\cos\left(\frac{n\pi x}{2}\right) = -\frac{2}{n\pi} x \cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^2 + \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx = (-1)^{n+1}\frac{4}{n\pi}.$$

7. *(30 pts.)* Let $\mathcal{V}$ be the vector space of quadratic polynomials (called $\mathcal{P}_2$ by Fulling but $\mathcal{P}_3$ by Leon). Let $L\colon \mathcal{V} \to \mathcal{V}$ be the differential operator $(Lp)(t) \equiv p''(t) + 3tp'(t)$.

(a) Find the matrix that represents L with respect to the standard basis $\{t^2, t, 1\}$ for $\mathcal{V}$.

$$L(t^2) = 2 + 6t^2, \qquad L(t) = 3t, \qquad L(1) = 0.$$

Put the coefficients into the columns of the matrix: $M = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{pmatrix}$.

(b) Find the kernel of L.

Row reduce M: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The kernel of M is the vectors of form $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, which means that the kernel of L is the constant functions.

(c) Find the range of L.

Transpose M and reduce again:

$$\begin{pmatrix} 6 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Interpret each row as a polynomial: A basis for the range is $\left\{t^2 + \frac{1}{3}, t\right\}$.

(d) Use "superposition principles" to find all polynomial solutions in $\mathcal{V}$ of the differential equation $p''(t) + 3tp'(t) = t$.

We see that t is in the range, so solutions exist. In fact, from the calculation in (a) we see that $L(t/3) = t$. To this we must add the most general solution of the homogeneous equation, which represents the kernel found in (b). So the general solution of the nonhomogeneous equation is

$$\frac{t}{3} + c \qquad (c \text{ arbitrary}).$$

8. *(30 pts.)* The vibrations of a violin string are determined by the wave equation $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$. The string is "tied down" at its ends, $x = 0$ and $x = \pi$, so u equals 0 there. The initial data are $u(0, x) = f(x)$ and $\frac{\partial u}{\partial t}(0, x) = g(x)$.

Solve the problem by separation of variables. (Of course, you won't be able to evaluate the integrals for the Fourier coefficients, but you can write down the formulas.)

Step 1: Find separated solutions, $u = X(x)Y(y)$.

$$\frac{X''}{X} = \frac{Y''}{Y} = -\lambda^2.$$

X must satisfy the homogeneous boundary conditions at the ends, so

$$X(x) = \sin(\lambda x) \quad \text{and} \quad \lambda = n \text{ (integer)}.$$

$$T(t) = A\cos(nt) + B\sin(nt).$$

Step 2: Form a series that satisfies the nonhomogeneous data.

$$u(t, x) = \sum_{n=1}^{\infty} [A_n \cos(nt)\sin(nx) + B_n \sin(nt)\sin(nx)]$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(nx), \qquad g(x) = \sum_{n=1}^{\infty} nB_n \sin(nx).$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx) f(x)\, dx, \qquad B_n = \frac{2}{n\pi}\int_0^{\pi} \sin(nx) g(x)\, dx.$$

The last 5 points are free!