Math. 309 (Fulling) 25 September 2020

Test A – Solutions

Upload your answers, in order, as a single document.

Calculators may be used for simple arithmetic operations only!

1. *(20 pts.)* Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 0 \end{pmatrix}$.

Calculate each of these, or declare it undefined:

(a) $A + B$

undefined

(b) AB

$\begin{pmatrix} 7 & 2 & -2 \\ 1 & 1 & 4 \end{pmatrix}$

(c) BA

undefined

(d) A^{-1}

$\frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

(e) B^{-1}

undefined

2. *(20 pts.)* Find all solutions of each of these systems.

(a) $\begin{cases} x + 2y = 2, \\ -2x + y = 1. \end{cases}$

$$\begin{pmatrix} 1 & 2 & 2 \\ -2 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 \\ 0 & 5 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Therefore, $y = 1$ and $x = 0$.

Alternative solution: Use Qu. 1(d):

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b) $\begin{cases} x + 2y - 3z = 0, \\ 2x + y - z = 3, \\ x - y + 2z = 3. \end{cases}$

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 5 & 3 \\ 0 & -3 & 5 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \frac{1}{3} & 2 \\ 0 & 1 & -\frac{5}{3} & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore,

$$z \text{ is arbitrary}, \quad y = \frac{5z}{3} - 1, \quad x = -\frac{z}{3} + 2.$$

3. *(21 pts.)* Determine whether each set is linearly independent. If it is not, find an independent set with the same span.

(a) $$\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

independent

(b) $$\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \right\}$$

Write the vectors as rows:

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & -5 \\ 0 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix}.$$

Therefore, the vectors are independent.

Alternative solution: Take the determinant of the matrix (or its transpose):

$$\begin{vmatrix} 1 & 0 & 3 \\ 2 & -1 & 1 \\ 0 & -1 & -1 \end{vmatrix} = 1 \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 3 \\ -1 & -1 \end{vmatrix} = 2 - 2 \times 3 = -4 \neq 0,$$

so the vectors are independent.

(c) $\{t^2 - t, \quad t+1, \quad 5t^2 - 4t + 1\}$

Here the vectors already appear as rows.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 5 & -4 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the vectors are dependent, and a basis for their span is $\{t^2 - t, t+1\}$.

There are other correct answers for the spanning basis. In particular, if you carry the Gauss–Jordan reduction out to the end, you get

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which prescribes the basis $\{t^2 + 1, t + 1\}$.

4. *(10 pts.)* Calculate the determinant $\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$.

Expanding in cofactors of either the first row or the first column, we get

$$2 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 2 \times 3 - 1 - 1 = 4.$$

5. *(Essay – 10 pts.)* The set of all 3×3 matrices (with real entries) is a vector space, given the usual definitions of addition and scalar multiplication of matrices. What is its dimension? Explain.

The dimension is 9. I don't put example essays in test keys.

6. *(19 pts.)* Find the inverse (if it exists) of the matrix $M = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix}$.

Indicate what row operations you're performing.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & -\frac{3}{2} & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -\frac{3}{2} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 2 & -\frac{3}{2} & 1 \\ 0 & 0 & 1 & -1 & \frac{3}{2} & -1 \end{pmatrix}.$$

Therefore,

$$M^{-1} = \begin{pmatrix} -1 & \frac{1}{2} & 0 \\ 2 & -\frac{3}{2} & 1 \\ -1 & \frac{3}{2} & -1 \end{pmatrix}.$$