Math. 309 (Fulling) 30 October 2020

Test B – Solutions

Upload your answers, in order, as a single document.

Calculators may be used for simple arithmetic operations only!

1. *(16 pts.)* Classify each of these equations as linear homogeneous, linear nonhomogeneous, or nonlinear.

(a) $x - 2y + z - 1 = 0$ (to be solved for (x, y, z))

linear nonhomogeneous

(b) $\dfrac{dy}{dt} + t^2\, y = 0$ (to be solved for $y(t)$)

linear homogeneous

(c) $\displaystyle\int_0^1 e^{2(x-t)} f(t)\, dt + f(x) = 0$ (to be solved for f , a function of x)

linear homogeneous

(d) $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial t} + e^u = 0$ (to be solved for $u(x,t)$)

nonlinear

2. *(24 pts.)*

(a) Find all the eigenvalues and eigenvectors of $M = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$.

Eigenvalues: $0 = \begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = (\lambda - 3)(\lambda + 2) + 4 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$. Hence the eigenvalues are 2 and -1.

Eigenvectors for $\lambda = 2$ *:* (Suppress the column of zeros at the right.)

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}.$$

Thus the eigenvectors $\binom{x}{y}$ satisfy $x - 2y = 0$, or $x = 2y$. A nice basis eigenvector is $\vec{b}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Eigenvectors for $\lambda = -1$ *:*

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}.$$

Thus the eigenvectors satisfy $2x - y = 0$, or $y = 2x$. A nice basis eigenvector is $\vec{b}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(b) Use the results of (a) to solve the ODE system $\dfrac{dx}{dt} = 3x - 2y, \quad \dfrac{dy}{dt} = 2x - 2y,$ with arbitrary initial data $x(0)$, $y(0)$.

Method 1: There is a basis of solutions of the form $\vec{b}_j e^{\lambda_j t}$. Thus the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}.$$

Then solve

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2c_1 + c_2 \\ c_1 + 2c_2 \end{pmatrix}$$

for the coefficients: $c_1 = \frac{2}{3}x(0) - \frac{1}{3}y(0), \quad c_2 = -\frac{1}{3}x(0) + \frac{2}{3}y(0)$.

Method 2: Let $D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ and $U = (\, \vec{b}_1 \quad \vec{b}_2 \,) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Then $M = UDU^{-1}$ and

$$e^{Mt} = Ue^{Dt}U^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \left(\frac{1}{3}\right)$$

$$= \frac{1}{3} \begin{pmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4e^{2t} - e^{-t} & -2e^{-2t} + 2e^{-t} \\ 2e^{2t} - 2e^{-t} & -e^{2t} + 4e^{-t} \end{pmatrix}.$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{Mt} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

The two methods agree. Notice that the effect of U^{-1} is just to solve the equations in the other method that relate c_1 and c_2 to $x(0)$ and $y(0)$.

3. *(26 pts.)* The linear function $K\colon \mathbf{R}^3 \to \mathbf{R}^3$ is represented by the matrix

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{pmatrix}.$$

(a) Find the kernel of K .

Find the eigenvectors with eigenvalue 0 . (Again omit the column of zeros at the right.)

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the kernel contains only the zero vector.

(b) Find the range of K.

We want a basis for the column space of B, which we get by row-reducing the transpose of B:

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the range is all of $\mathbf{R}^3$. We should have known this already from part (a):

$$\dim \operatorname{ran} = \dim \operatorname{dom} - \dim \ker = 3 - 0 = 3\,.$$

(c) Is K injective? surjective? bijective? (List all the terms that apply.)

All three apply.

4. *(16 pts.)* The linear function $L\colon \mathbf{R}^2 \to \mathbf{R}^2$ is represented by the matrix $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ with respect to the natural basis. Find the matrix representing L with respect to the basis

$$\left\{ \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

(Change the basis in both domain and codomain.)

Form a matrix out of the new basis vectors:

$$U = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \qquad U^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}.$$

U maps $\vec{v}$-basis coordinates to natural coordinates, so the matrix we need is

$$\tilde{A} \equiv U^{-1}AU = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 5 & -4 \\ 4 & -5 \end{pmatrix}.$$

As a partial check, note that the trace and determinant are unchanged by the similarity transformation:

$$\operatorname{tr} \tilde{A} = 0 = \operatorname{tr} A, \qquad \det \tilde{A} = -1 = \det A\,.$$

(This is necessary, but not sufficient, for our arithmetic to be correct.)

5. *(18 pts.)* Find an orthonormal basis for the span of

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} \sqrt{3} \\ -\sqrt{3} \\ 0 \\ 0 \end{pmatrix},$$

by applying the Gram–Schmidt process to these vectors in the order given.

The normalized first vector is $\hat{u}_1 = \frac{1}{2}(1,1,1,1)^{\mathrm{t}}$. Then

$$\vec{v}_2 \cdot \hat{u}_1 = \frac{1}{2}\,, \qquad \vec{v}_{2\|} = (\vec{v}_2 \cdot \hat{u}_1)\hat{u}_1 = \frac{1}{4}(1,1,1,1)^{\mathrm{t}}, \qquad \vec{v}_{2\perp} = \vec{v}_2 - \vec{v}_{2\|} = \frac{1}{4}(3,-1,-1,-1)^{\mathrm{t}}.$$

Normalize:

$$\|\vec{v}_{2\perp}\|^2 = \frac{12}{16} = \frac{3}{4}\,, \qquad \hat{u}_2 = \frac{1}{2\sqrt{3}}(3,-1,-1,-1)^{\mathrm{t}}.$$

Finally, $\vec{v}_3 \cdot \hat{u}_1 = 0$, $\vec{v}_3 \cdot \hat{u}_2 = \frac{3\sqrt{3}+\sqrt{3}}{2\sqrt{3}} = 2$, so

$$\vec{v}_{3\|} = (\vec{v}_3 \cdot \hat{u}_2)\hat{u}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \qquad \vec{v}_{3\perp} = \vec{v}_3 - \vec{v}_{3\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}.$$

Normalize: $\|\vec{v}_{3\perp}\|^2 = 2$, $\hat{u}_3 = \frac{1}{\sqrt{6}}(0,-2,1,1)^{\mathrm{t}}$.

Summary:

$$\hat{u}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \hat{u}_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \qquad \hat{u}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}.$$

To check, I row-reduced each basis and got

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

both times, so the spans are the same.