Math. 311 (Fulling) 15 December 1998

Final Exam – Solutions

1. *(15 pts.)* For each of these vector fields, determine whether a potential energy function exists (so that $\vec{F} = -\nabla V$ everywhere in $\mathbf{R}^3$).

   (a) $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$.

YES. $\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = 0$.

   (b) $\vec{F} = z\,\hat{\jmath} - y\,\hat{k}$.

NO. $\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & z & -y \end{vmatrix} = -2\,\hat{\imath} \neq 0$.

   (c) $\vec{F} = -\dfrac{y}{x^2+y^2}\,\hat{\imath} + \dfrac{x}{x^2+y^2}\,\hat{\jmath}$.

NO. This is the famous example of a field whose integral around certain closed paths is nonzero, even though $\nabla \times \vec{F} = 0$ everywhere except at the origin (where it is undefined). See the last subsection of Sec. 7.5.

2. *(15 pts.)* The Rapidrudder polynomials are the normalized orthogonal polynomials defined by applying the Gram–Schmidt algorithm to the power functions, with respect to the inner product

$$\langle p, q \rangle = \int_1^4 p(t)q(t)\,dt.$$

   Find the first 2 Rapidrudder polynomials.

Call the orthonormal polynomials $\hat{u}_0(t)$, $\hat{u}_1(t)$, ... We calculate

$$\|1\|^2 = \int_1^4 1\,dt = 3,$$

so the first polynomial is

$$\hat{u}_0(t) = \frac{1}{\sqrt{3}}\,.$$

The projection of the next power function onto this is

$$t_{\|} = \langle \hat{u}_0, t \rangle \hat{u}_0 = \frac{1}{3}\int_1^4 t\,dt = \frac{1}{6}(16-1) = \frac{5}{2}\,.$$

Therefore,

$$t_{\perp} = t - \frac{5}{2}\,.$$

Then

$$\begin{aligned}\|t_\perp\|^2 &= \int_1^4 \left(t^2 - 5t + \frac{25}{4}\right) dt \\ &= \left[\frac{t^3}{3} - \frac{5t^2}{2} + \frac{25t}{4}\right]_1^4 \\ &= \frac{63\cdot 4 - 75\cdot 6 + 75\cdot 3}{12} = \frac{27}{12} = \frac{9}{4}\,.\end{aligned}$$

Thus

$$\hat{u}_1(t) = \frac{2}{3}\left(t - \frac{5}{2}\right).$$

3. *(20 pts.)* A linear function $L\colon \mathbf{R}^4 \to \mathbf{R}^4$ is represented by the matrix $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \\ 3 & 2 & 3 & -1 \\ 0 & 2 & 0 & -1 \end{pmatrix}$.

(a) Find a basis for the range of L.

Transpose and row-reduce:

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 2 & 2 \\ 1 & 2 & 3 & 0 \\ 0 & -1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, a basis is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$. Of course, the answer is not unique, in either this part or the next.

(b) Find a basis for the kernel of L.

Row-reduce and solve:

$$\to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus (if the unknowns are lettered (w, x, y, z)) the solutions of the homogeneous equation satisfy

$$y \text{ and } z \text{ arbitrary}, \qquad w = -y, \qquad x = \frac{z}{2}\,.$$

Therefore, a basis is $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\}$.

4. *(20 pts.)*

(a) Define "subspace".

A subspace is a subset of a vector space that is closed under addition and scalar multiplication. (The meaning of these terms is made clear by the examples in part (b).)

(b) Prove that one of these is a subspace (of a vector space $\mathcal{V}$). [Do either (A) or (B) – your choice.]

(A) The kernel of a linear function $L\colon \mathcal{V} \to \mathcal{V}$.

If $L(\vec{u}_1) = 0 = L(\vec{u}_2)$, then $L(r\vec{u}_1 + \vec{u}_2) = rL(\vec{u}_1) + L(\vec{u}_2) = 0$. That is, if $\vec{u}_1$ and $\vec{u}_2$ are in the kernel of L , then so are their linear combinations.

(B) The span of a list of vectors in $\mathcal{V}$, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N\}$.

If $\vec{u}_1 = \sum_{j=1}^N c_j \vec{v}_j$ and $\vec{u}_2 = \sum_{j=1}^N d_j \vec{v}_j$, then

$$r\vec{u}_1 + \vec{u}_2 = \sum_{j=1}^N (rc_j + d_j)\vec{v}_j \,.$$

That is, linear combinations of linear combinations are themselves linear combinations of the original vectors.

5. *(20 pts.)* Let $u = e^x \cos y$, $v = e^x \sin y$. Note that when $\binom{x}{y} = \binom{0}{\pi}$, $\binom{u}{v} = \binom{-1}{0}$.

(a) Find the first-order ("best affine") approximation to u and v when $\binom{x}{y} = \binom{0.10}{3.13}$.

The Jacobian matrix, evaluated at the base point, is

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore (since $3.13 - \pi = -0.01$ to the precision indicated),

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} &\approx \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + J \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0.10 \\ -0.01 \end{pmatrix} = \begin{pmatrix} -1.10 \\ 0.01 \end{pmatrix}. \end{aligned}$$

(b) How would you approximately find the change in x and y when $\binom{u}{v}$ moves slightly away from $\binom{-1}{0}$?

By the inverse function theorem,

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = J^{-1} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

(where $\Delta u = u - (-1)$, $\Delta v = v - 0$). In the present case, $J^{-1} = J$, given above.

6. *(20 pts.)* Find the flux of the vector field $\vec{B} = \hat{\imath} + z\,\hat{k}$ through the unit hemisphere $x^2 + y^2 + z^2 = 1$, $z > 0$. The choice of method is up to you.

Method 1: Note that $\nabla \cdot \vec{B} = 1$. We can apply Gauss's theorem if we turn the hemisphere into a closed surface by adding a flat disk at the bottom. On that disk, $\vec{B} \cdot \hat{n} = \vec{B} \cdot \hat{k} = 0$, because $z = 0$ there. Therefore, the flux through the hemisphere equals the integral of $\nabla \cdot \vec{B} = 1$ over the half-ball ("solid hemisphere"), which is just the volume of the half-ball,

$$\frac{1}{2} \times \frac{4}{3}\pi r^3 = \frac{2}{3}\pi.$$

Method 2: The unit normal vector to the sphere is $\hat{n} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$, where $r = \sqrt{x^2 + y^2 + z^2} = 1$ in this case. The element of surface area on the sphere is $r^2 \sin\theta\, d\theta\, d\phi$ (where θ is the polar angle and ϕ is the azimuthal angle.) Therefore, $\hat{n} \cdot \vec{B} = x + z^2$, and the flux is

$$\int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi\, \sin\theta(\sin\theta\cos\phi + \cos^2\theta).$$

The integral over ϕ in the first term is zero. The second term becomes

$$2\pi \int_0^{\pi/2} \sin\theta\cos^2\theta\, d\theta = 2\pi(-1)\left.\frac{\cos^3\theta}{3}\right|_0^{\pi/2} = \frac{2}{3}\pi.$$

Method 3: Do a full-scale parametrization, following Example 2 in Sec. 7.6.

7. *(10 pts.)* We know that the determinant of a diagonal or triangular matrix is the product of the elements on the main diagonal. Is the corresponding thing true of a "backwards diagonal" matrix, such as $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$? Justify your answer completely, being careful about *minus signs* and the possible difference between odd and even dimensions!

Oops! I got this wrong the first time. (I thought the answer was always "yes", but for different reasons in even and odd dimensions.) The question is much more subtle than I thought, so I graded it very leniently.

If the dimension is 2, the determinant is the *negative* of the backward diagonal elements. If the dimension is 3, the determinant is a corner element, times a checkerboard sign of $+$, times a 2×2 of the type we just discussed, so again there is an overall minus sign. If the dimension is 4, the determinant is a corner element with a checkerboard sign of $-$, times a 3×3 of the type just discussed, so the net sign is positive. If the dimension is 5, the new checkerboard sign is $+$, so the overall sign is still positive. Continuing in this way, we see that the sign changes whenever we hit an even dimension. In summary, therefore,

- The determinant is the product of the backward-diagonal elements if the dimension is 1, 4, 5, 8, 9, … . (For graduates of Math. 302, these are the integers equal to either 0 or 1 modulo 4.)
- The determinant is the negative of the product of the backward diagonal elements if the dimension is 2, 3, 6, 7, … . (These are the integers equal to 2 or 3 modulo 4.)

Another proof: We can convert the matrix to an ordinary diagonal matrix by completely reversing the order of the rows. The number of interchanges of adjacent rows required for this is $(n-1) + (n-2) + \cdots = \frac{1}{2}n(n-1)$ (the same as the number of independent elements in an antisymmetric $n \times n$ matrix). This number is even if n or $n-1$ is divisible by 4, and odd otherwise.

8. *(20 pts.)* The only eigenvalues of the matrix $M = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ are $\lambda = 2$ and 4 .

   Find a basis for $\mathbf{R}^3$ consisting of eigenvectors of M ; make the basis orthonormal, or else explain why that is impossible.

Subtract λ from each diagonal element and solve the resulting homogeneous equations.

$$\lambda = 2: \qquad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x - z = 0, \\ y \text{ arbitrary.} \end{array}$$

Thus a normalized basis is $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

$$\lambda = 4: \qquad \begin{pmatrix} -1 & 1 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{r} x + z = 0, \\ y = 0. \end{array}$$

Thus a normalized eigenvector is $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. It is automatically orthogonal to the other eigenvectors, because the matrix is symmetric.

9. *(20 pts.)* Roger Rapidrudder has just learned that his position, relative to the IQGR Radar Positioning Network, is 3 miles west of where he thought he was. Therefore, Roger needs to convert a set of data in $\mathcal{P}_2$ from the basis $\{\vec a_1 = t^2, \vec a_2 = t, \vec a_3 = 1\}$ to the basis $\{\vec b_1 = (t+3)^2, \vec b_2 = t+3, \vec b_3 = 1\}$.

   (a) Find the matrix U that will convert the coordinates of an arbitrary element $p \in \mathcal{P}_2$ with respect to the $\mathbf{a}$ basis into the coordinates of p with respect to the $\mathbf{b}$ basis.

Method 1: We have

$$\begin{aligned} \vec b_1 &= \vec a_1 + 6\vec a_2 + 9\vec a_3 , \\ \vec b_2 &= \phantom{\vec a_1 + {}} \vec a_2 + 3\vec a_3 , \\ \vec b_3 &= \vec a_3 . \end{aligned}$$

So the matrix that maps the old basis to the new basis is $Q = \begin{pmatrix} 1 & 6 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$. The matrix that maps the old coordinates to the new coordinates is the inverse of the transpose of this. So we need to row-reduce

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 6 & 1 & 0 & 0 & 1 & 0 \\ 9 & 3 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -6 & 1 & 0 \\ 0 & 0 & 1 & 9 & -3 & 1 \end{pmatrix} .$$

Thus

$$U = (Q^{\mathrm{t}})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -6 & 1 & 0 \\ 9 & -3 & 1 \end{pmatrix}.$$

Method 2: Suppose the original polynomial is $at^2 + bt + c$. This is $at^2 + b(t+3) - 3b + c$. Can we handle the t^2 term in the same elementary way? Yes, if we are careful to subtract whatever we add, as we "complete the square":

$$\begin{aligned} at^2 + bt + c &= a(t^2 + 6t + 9) - 6at - 9a + b(t+3) - 3b + c \\ &= a(t+3)^2 + (b - 6a)(t+3) + 6 \cdot 3a - 9a - 3b + c \\ &= a(t+3)^2 + (b - 6a)(t+3) + (9a - 3b + c). \end{aligned}$$

Thus we read off

$$U = \begin{pmatrix} 1 & 0 & 0 \\ -6 & 1 & 0 \\ 9 & -3 & 1 \end{pmatrix}.$$

Method 3: Like Method 2, but working in the other direction. This is simpler and more natural than Method 2, but it leaves you with the task of calculating the inverse matrix at the end.

(b) If a linear function $L\colon \mathcal{P}_2 \to \mathcal{P}_2$ has the matrix M with respect to the old (**a**) basis, what matrix represents L in Roger's new coordinate system? Give the answer symbolically (but unambiguously); *don't* work out the arithmetic.

$$UMU^{-1}$$

10. *(25 pts.)*

(a) Find the eigenvalues and eigenvectors of the matrix $M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

Eigenvalues:

$$0 = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1).$$

Thus $\lambda = 3$ or 1 .

$\lambda = 3:$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow x - y = 0.$$

Thus the eigenvectors are the multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\lambda = -1:$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow x + y = 0.$$

Thus the eigenvectors are the multiples of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(b) Explain how to use the result of (a) to solve the differential equation system

$$\frac{dx}{dt} = x + 2y, \qquad x(0) = -2,$$
$$\frac{dy}{dt} = 2x + y, \qquad y(0) = 5.$$

Extra credit if you can actually carry out all the calculations!

The system can be written in the vectorial form

$$\frac{d\vec{y}}{dt} = M\vec{y}, \qquad \vec{y}(0) = \begin{pmatrix} -2 \\ 5 \end{pmatrix}.$$

The solution is

$$\vec{y}(t) = e^{tM}\vec{y}(0).$$

The solution matrix can be constructed as

$$e^{tM} = Ue^{tD}U^{-1},$$

where D is the diagonalized form of M :

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e^{tD} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix};$$

and U is the matrix mapping coordinates with respect to the eigenbasis into coordinates with respect to the natural basis, constructed out of the eigenvectors:

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad U^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(Of course, the eigenvalues can be put into D in the other order, provided that the the eigenvectors are reversed in U likewise.) After algebra and simplification, one gets

$$e^{tM} = \frac{1}{2}\begin{pmatrix} e^{3t} + e^{-t} & e^{3t} - e^{-t} \\ e^{3t} - e^{-t} & e^{3t} + e^{-t} \end{pmatrix}$$

and hence

$$x = \tfrac{1}{2}(3e^{3t} - 7e^{-t}), \qquad y = \tfrac{1}{2}(3e^{3t} + 7e^{-t}).$$

A further calculation verifies that these satisfy the differential equations and the initial conditions.

11. **Holiday Bonus: 15 free points to bring the total up to 200!**

12. *(10 extra credit pts.)* Here are some parts I left out to make the test shorter. Do **ONE** of them for extra points.

   (A) Evaluate the surface integral in Qu. 6 by a (substantially!) different method.

   (B) Prove the other theorem in Qu. 4(b).

   (C) *Do* work out the arithmetic in Qu. 9(b).

   (D) Find the potential energy for a case where you answered "yes" in Qu. 1.

   (E) Suppose that the matrices in Qu. 8 and Qu. 10 are Hessians (the matrices of second-order partial derivatives at critical points of scalar-valued functions). Determine in each case whether the critical point marks a maximum, a minimum, or a saddle point.