Math. 311 (Fulling) 20 November 1998

Test C – Solutions

Name: ______________________________ Number: ______________
(as on attendance sheets)

1. *(20 pts.)*

(a) Find the volume of the parallelepiped determined by the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

$$\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 3 \\ -1 & -1 & 3 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -1 & -1 & 1 \end{vmatrix} = \underset{\text{[cofactors of first column]}}{3\left[\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}\right]}$$

$$= 3[(2+1) - (1-2)] = 3 \cdot 4 = 12.$$

Since this is positive, it is the volume.

(b) Calculate the determinant $\begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 3 \\ 0 & 0 & 2 & 3 \\ -1 & 5 & -1 & 3 \end{vmatrix}$.

Method 1: Cofactors of first row.

$$-\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 3 \\ -1 & -1 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ -1 & 5 & -1 \end{vmatrix} = -12 \quad \text{[from (a)]} \quad - (-1) \cdot 2 \begin{vmatrix} 1 & 0 \\ -1 & 5 \end{vmatrix} \quad \text{[from 2nd row]}$$

$$= -12 + 10 = -2.$$

Method 2: Add row 2 to row 4, and exchange rows 1 and 2. Then subtract 5 times new row 2 from row 4:

$$-\begin{vmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 5 & 0 & 0 \end{vmatrix} \to -\begin{vmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1 \cdot 1 \cdot 2 \cdot 1 = -2.$$

2. *(30 pts.)* Introduce a coordinate system (u, v) into $\mathbf{R}^2$ by $\begin{cases} x = u + v^2, \\ y = \frac{1}{2}v. \end{cases}$

(a) Find the basis of tangent vectors to the coordinate curves at each point.

$$\frac{\partial \vec{r}}{\partial u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{(tangent to curves of constant } v\text{)},$$
$$\frac{\partial \vec{r}}{\partial v} = \begin{pmatrix} 2v \\ \frac{1}{2} \end{pmatrix} \quad \text{(tangent to curves of constant } u\text{)}.$$

These are the columns of the Jacobian matrix, J (used in parts (b) and (c)). For use in part (d), we note that at $u = -1$, $v = 2$,

$$\frac{\partial \vec{r}}{\partial u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \frac{\partial \vec{r}}{\partial v} = \begin{pmatrix} 4 \\ \frac{1}{2} \end{pmatrix}.$$

(b) Calculate the basis of normal vectors to the coordinate hypersurfaces (which are curves in this 2-dimensional situation) at the point where $u = 1$, $v = 2$.

These are the rows of J^{-1}. We can set $v = 2$ before inverting:

$$\begin{pmatrix} 1 & 4 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1} = 2 \begin{pmatrix} \frac{1}{2} & -4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -8 \\ 0 & 2 \end{pmatrix}$$

(since the determinant of J is $\frac{1}{2}$). Therefore,

$$\nabla u = (1, -8) \quad (\perp \text{ to “surface” of constant } u),$$
$$\nabla v = (0, 2) \quad (\perp \text{ to “surface” of constant } v).$$

(c) Calculate the double integral $\int_D (x + y^2)\, dx\, dy$ when D is the region bounded by the curves

$$v = 0, \qquad v = 2, \qquad u = 1, \qquad u = -1.$$

The Jacobian determinant is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 2v \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

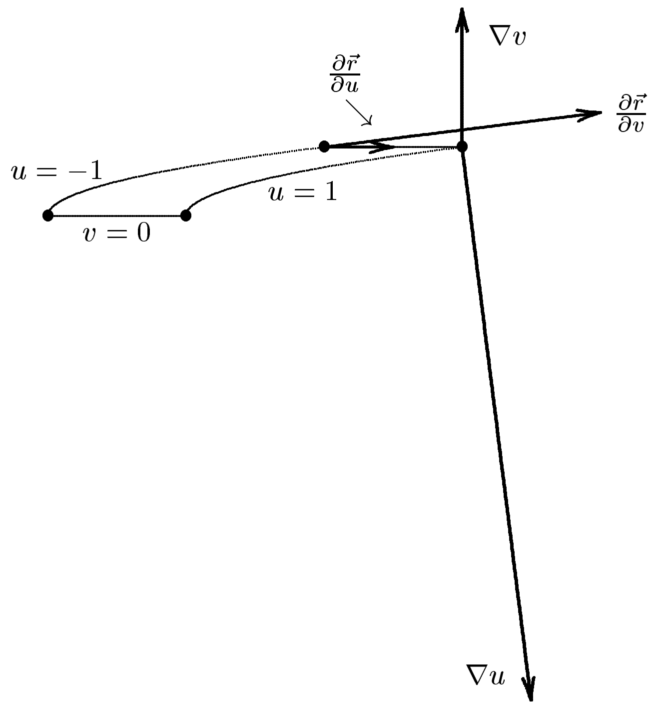
Therefore, the integral equals

$$\int_0^2 dv \int_{-1}^1 du \left(\tfrac{1}{2}\right) \left[(u + v^2) + \frac{v^2}{4}\right] = \frac{1}{2}\left[2\int_{-1}^1 u\, du + 2\int_0^2 \frac{5}{4} v^2\, dv\right]$$

$$= \frac{u^2}{2}\Big|_{-1}^{1} + \frac{5}{4}\frac{v^3}{3}\Big|_0^2 = \frac{5 \cdot 8}{12} = \frac{10}{3}.$$

(d) Sketch the region D, the basis of tangent vectors at the point $u=-1$, $v=2$, and the basis of normal vectors at the point $u=1$, $v=2$.

Curve $v=0$: $x=u$, $y=0$ (horizontal line segment).
Curve $v=2$: $x=u+4$, $y=1$ (horizontal line segment).
Curve $u=1$: $x=1+v^2$, $y=\frac{v}{2} \Rightarrow x=1+4y^2$ (parabola).
Curve $u=-1$: $x=-1+v^2$, $y=\frac{v}{2} \Rightarrow x=-1+4y^2$ (parabola).



3. *(12 pts.)* Prove the identity $\nabla\cdot(\nabla\times\vec A)=0$.

$$\nabla\cdot\begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ \partial_x & \partial_y & \partial_z\\ A_x & A_y & A_z\end{vmatrix} = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

$$=\frac{\partial^2 A_z}{\partial x\,\partial y}-\frac{\partial^2 A_z}{\partial y\,\partial x}-\frac{\partial^2 A_y}{\partial x\,\partial z}+\frac{\partial^2 A_y}{\partial z\,\partial x}+\frac{\partial^2 A_x}{\partial y\,\partial z}-\frac{\partial^2 A_x}{\partial z\,\partial y}=0+0+0=0.$$

Alternative argument: Using algebraic properties of the cross product,

$$\nabla\cdot(\nabla\times\vec A)=\vec A\cdot(\nabla\times\nabla)=0.$$

This equation must not be taken literally, because ∇ is not an ordinary vector, but a differential operator. However, if it is understood that all the partial derivatives continue to act upon $\vec A$, then the equation accurately reflects the algebraic structure of the expression and the conclusion is correct.

4. *(13 pts.)* Evaluate $\int_S \vec{F}\cdot d\vec{S}$ when $\vec{F} = (y-2z)\hat{\imath} + 2z\hat{\jmath} + \hat{k}$ and S is the patch of surface (with upward-pointing normal) parametrized by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = (400-r^2)(10+\sin r)(2+\cos(5\theta));$$
$$0 \le r < 20, \qquad 0 < \theta < 2\pi.$$

Hint: The boundary of S is a circle in the (x,y) coordinate plane ($z=0$).

The boundary is the circle $r = 20$. The first thing to notice is that

$$\nabla\cdot\vec{F} = 0+0+0 = 0,$$

so we can smash the surface flat into the plane with impunity. (The arguments for this are reviewed below.) It is now the disk $0 \le r < 20$, $z = 0$. Therefore, the integral becomes

$$\int_0^{20} r\,dr \int_0^{2\pi} d\theta\,(\vec{F}\cdot\hat{k}) = \text{Area of disk [because } \vec{F}\cdot\hat{k} = 1\text{]} = \pi(20)^2 = 400\pi.$$

There are two arguments for why the circle can be moved. (1) Since $\nabla\cdot\vec{F} = 0$, the flux integral of $\vec{F}$ over any *closed* surface is zero (Gauss's theorem). The easiest way to close up our surface is to add a disk at the bottom. Then Gauss says that the integral over the original surface must cancel the integral over the disk with the *downward* pointing normal vector. In other words, it is equal to the disk integral with the upward pointing normal vector, $+\hat{k}$. (2) Since $\nabla\cdot\vec{F} = 0$ *everywhere* in space (no holes), we are assured that there exists a vector field $\vec{A}$ such that $\vec{F} = \nabla\times\vec{A}$. We do not need to know what $\vec{A}$ is! The important fact is that the flux integral of $\vec{F}$ through S is equal to the line integral of $\vec{A}$ around the boundary circle. Clearly, this is the same for *any* surface with that circle as boundary, so again we see that the flux through the complicated surface must equal the flux through the disk, which is much easier to calculate.

5. *(25 pts.)* Consider $\mathbf{R}^4$ with the usual inner product.

   (a) Find an orthonormal basis for the span of the vectors

$$\vec{v}_1 = (1,0,0,0), \qquad \vec{v}_2 = (0,0,1,1), \qquad \vec{v}_3 = (1,1,1,2).$$

$\vec{v}_1$ is already of unit length, and $\vec{v}_2$ is already orthogonal to $\vec{v}_1$, so we can set

$$\hat{u}_1 = \vec{v}_1\,, \qquad \hat{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{2}}(0,0,1,1).$$

The part of $\vec{v}_3$ lying in the span of these first two vectors is

$$\vec{v}_{3\|} = \langle\hat{u}_1,\vec{v}_3\rangle\hat{u}_1 + \langle\hat{u}_2,\vec{v}_3\rangle\hat{u}_2 = 1(1,0,0,0) + \frac{1}{\sqrt{2}}(3)(0,0,1,1)\frac{1}{\sqrt{2}} = \left(1,0,\tfrac{3}{2},\tfrac{3}{2}\right).$$

So the perpendicular part of $\vec{v}_3$ is

$$\vec{v}_\perp = \vec{v}_3 - \vec{v}_{3\|} = \left(0,1,-\tfrac{1}{2},\tfrac{1}{2}\right);$$
$$\|\vec{v}_{3\perp}\|^2 = 1 + \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{3}{2}\,,$$

so

$$\hat{u}_3 = \sqrt{\tfrac{2}{3}}\left(0,1,-\tfrac{1}{2}.\tfrac{1}{2}\right) = \left(0,\sqrt{\tfrac{3}{2}},-\tfrac{1}{\sqrt{6}},\tfrac{1}{\sqrt{6}}\right) \quad\text{or}\quad \tfrac{1}{\sqrt{6}}(0,2,-1,1).$$

(b) Express $\vec{w} = (3, 1, -1, 0)$ as a linear combination of the orthonormal vectors you constructed in (a). (If you had trouble with (a) [or even if you didn't], explain in words what you should be doing here.)

We want to take the dot product of $\vec{w}$ with each basis vector, multiply by that vector, and add up:

$$\vec{w} = \langle \hat{u}_1, \vec{w} \rangle \hat{u}_1 + \langle \hat{u}_2, \vec{w} \rangle \hat{u}_2 + \langle \hat{u}_3, \vec{w} \rangle \hat{u}_3 = 3\hat{u}_1 + \frac{1}{\sqrt{2}}(-1)\hat{u}_2 + \left(\sqrt{\tfrac{2}{3}} + \tfrac{1}{\sqrt{6}}\right)\vec{u}_3 = 3\hat{u}_1 - \tfrac{1}{\sqrt{2}}\,\hat{u}_2 + \sqrt{\tfrac{3}{2}}\,\hat{u}_3 \,.$$

Notice that if $\vec{w}$ had been independent of the $\vec{v}$ vectors, this calculation would yield $\vec{w}_{\|}$, the projection of $\vec{w}$ onto the span of the vectors (the first step in calculating a fourth basis vector). In this case, however, $\vec{w}$ was chosen to lie already in the span of the $\vec{v}$ s, so that $\vec{w}_{\|} = \vec{w}$ and the calculation just yields its expansion in terms of the orthonormal basis.