1.2 Lines and Planes

When dealing with $\mathbf{R}^3$ or with physical space modeled by $\mathbf{R}^3$ (see the first two examples in Sec. 1.1), it is convenient to represent the different coordinates of a vector by different letters, thereby minimizing the use of subscripts:

$$\vec{x} = (x, y, z).$$

Often one deals with problems that are essentially two-dimensional; then the z component can be dropped:

$$\vec{x} = (x, y) \in \mathbf{R}^2.$$

Once the *origin* of a coordinate system in physical space has been fixed, each point in space can be identified with the vector with head at the point and tail at the origin.

As you know, a *(straight) line* in $\mathbf{R}^2$ can be defined by an equation of the form

$$ax + by = c, \tag{1}$$

where a, b, and c are constants. (Obviously, the equation is not unique, since all three constants can be multiplied by a nonzero number without changing the set of solutions (x, y).) Let us call this the *equation form* of a line. (If b is not zero, (1) can be rearranged into the "functional" form, $y = mx + d$.)

There is, however, another, equally good, way of representing a line. Introduce a new variable, t, which ranges through all the real numbers ($-\infty < t < \infty$). Consider (for example) the functions

$$x = 3t - 2, \qquad y = 2t.$$

Plotting the points (x, y) on graph paper for various values of t, one easily sees that they form a line. We can put the coordinates together in vectorial form:

$$\vec{x}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3t - 2 \\ 2t \end{pmatrix}. \tag{$*$}$$

Using the definitions of vector addition and multiplication, we can rewrite this as

$$\vec{x} = t \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

Thus each point on the line can be obtained by adding a fixed, or constant, vector to an arbitrary multiple of another fixed vector. The general case is

$$\vec{x} = t\vec{u} + \vec{x}_0 \tag{2}$$

(where vectors $\vec{u}$ and $\vec{x}_0$ are fixed, and t and $\vec{x}$ are variables). This is called the *parametric form* of a line (t being the "parameter" involved).

NOTATIONAL REMARK: In the foregoing paragraph we have begun the practice of writing vectors as *columns* of numbers instead of rows. This made the equation $(*)$ easier to read by separating the two coordinate formulas visually. Later (Secs. 2.4 and 3.2) we will encounter a more important reason for using columns, when a distinction will be made between two kinds of vectors, one written as columns and one as rows. On the other hand, column vectors are difficult to typeset and use up a lot of paper, so it is quite common to revert to the row notation when there is no danger of confusion.

Example 1. What is the equation form of the line $(*)$?

SOLUTION *(Method 1)*: Solve one of the coordinate equations for the parameter, and substitute into the other coordinate equation. Since $t = \frac{1}{2}y$, we have

$$x = \tfrac{3}{2}y - 2,$$

which can trivially be rearranged into the form (1).

Before giving a second method for solving this problem, we remark that if $\vec{x}_0$ in (2) is the *zero vector*,

$$\vec{0} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

then the line passes through the origin in $\mathbf{R}^2$. (On the other hand, if $\vec{x}_0 \neq \vec{0}$, the line may pass through the origin anyhow. See Exercise 1.2.11.) In this case the equation form is

$$0 = ax + by = \vec{a} \cdot \vec{x}.$$

That is, the vector $\vec{a} \equiv (a, b)$ is perpendicular* to all the vectors $\vec{x}$ making up the line. In the more general situation (1), the corresponding statement is that *$\vec{a}$ is perpendicular to the vector joining any two points on the line.*

PROOF: Let $\vec{x}_1 = (x_1, y_1)$ and $\vec{x}_2 = (x_2, y_2)$ be two points on the line. Then their coordinates satisfy

$$ax_1 + by_1 = c, \qquad ax_2 + by_2 = c.$$

Subtract these two equations to get

$$a(x_1 - x_2) + b(y_1 - y_2) = 0.$$

That is, $\vec{x}_1 - \vec{x}_2$ (the vector with head at $\vec{x}_1$ and tail at $\vec{x}_2$) is perpendicular to $\vec{a}$.

On the other hand, in the notation of (2) the difference between two vectors $\vec{x}_1$ and $\vec{x}_2$ on the line is always a multiple of $\vec{u}$. Thus $\vec{u}$ is a vector tangent to the line, and $\vec{u}$ *must be perpendicular to $\vec{a}$.* It follows (take the dot product of equation (2) with $\vec{a}$) that $\vec{a} \cdot \vec{x} = \vec{a} \cdot \vec{x}_0$, and therefore the c in (1) equals $\vec{a} \cdot \vec{x}_0$.

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* The words "perpendicular", "normal", and "orthogonal" are all equivalent when applied to vectors, lines, or planes. ("Orthogonal matrix" means something else, however, as will be explained in due course.)

Theorem 1: In $\mathbf{R}^2$ the vectors

$$\begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b \\ -a \end{pmatrix}$$

are perpendicular. (One is obtained by rotating the other through a right angle.)

This is the vectorial version of the familiar theorem, "Two lines are perpendicular if and only if the product of their slopes is -1."

SOLUTION TO EXAMPLE 1 *(Method 2)*: Since $\vec{u} = (3, 2)$, a vector perpendicular to it is $\vec{a} = (2, -3)$.† Then

$$\vec{a} \cdot \vec{x}_0 = (2, -3) \cdot (-2, 0) = -4,$$

so an equation for the line is

$$2x - 3y = -4.$$

(This is equivalent to the result of the first method.)

Example 2. Find a parametric representation of the line $x + 7y = 2$.

SOLUTION: Let $y = t$. Then $x = -7t + 2$. So the line is

$$\vec{x} = \begin{pmatrix} -7t + 2 \\ t \end{pmatrix} = t \begin{pmatrix} -7 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Note that there are many other correct answers, because our initial step was rather arbitrary. We could have taken $x = t$, or $y = 2t + 5$. The result would have been the same line, but with a different labeling of points by numbers t.

The parametric representation (2) makes sense in space of any dimension, not just in $\mathbf{R}^2$. (The same cannot be said for (1); we return to that problem later.)

Example 3. Find a parametric form of the line in $\mathbf{R}^3$ through the points $(3, 2, 4)$ and $(1, 1, -1)$.

SOLUTION: The vector difference between the two points, $(2, 1, 5)$, is parallel to the line and is therefore a suitable $\vec{u}$. The line passes through $(1, 1, -1)$, so that is a suitable $\vec{x}_0$. Therefore, we can write

$$\vec{x} = t(2, 1, 5) + (1, 1, -1),$$

or

$$x = 2t + 1, \qquad y = t + 1, \qquad z = 5t - 1.$$

To check, we see that we recover the two original points by taking $t = 1$ and $t = 0$.

The analogue of (1) for *planes* in $\mathbf{R}^3$ is

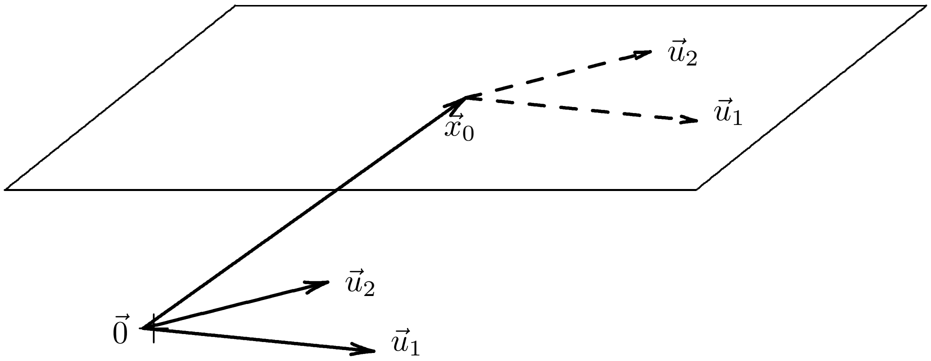
$$ax + by + cz = d. \tag{3}$$

† $(-2, 3)$ would do equally well in the role of $\vec{a}$. But you must use the same $\vec{a}$ on both sides of the equation $\vec{a} \cdot \vec{x} = \vec{a} \cdot \vec{x}_0$.

Planes also have parametric representations (which may be less familiar from elementary courses than the previous material in this section). Since a plane is a two-dimensional entity, one needs *two* independently varying parameters in order to sweep out all the points on it. It is not hard to guess that the correct analogue of (2) is

$$\vec{x} = s\vec{u}_1 + t\vec{u}_2 + \vec{x}_0\,, \tag{4}$$

where $\vec{x}_0$ is a point in the plane, and $\vec{u}_1$ and $\vec{u}_2$ are vectors tangent to the plane. We will demonstrate this with a sketch and with several examples.



Example 4. Find the parametric form of the plane

$$x - 3y + z = 0.$$

Solution: Choose $y = s$ and $z = t$ and solve for x as $3s - t$. Then

$$\vec{x} = \begin{pmatrix} 3s - t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Example 5. Find the equation form of the plane

$$\vec{x} = s(5, 0, -1) + t(2, 2, 0).$$

Solution: By analogy with the previous discussion of lines in $\mathbf{R}^2$, it is clear that the equation must be of the form $\vec{n} \cdot \vec{x} = 0$, where $\vec{n}$ is a vector perpendicular to the given $\vec{u}_1 = (5, 0, -1)$ and $\vec{u}_2 = (2, 2, 0)$. The main problem is finding such a vector.

Method 1: With the notation of (3) in mind, let $\vec{n} = (a, b, c)$. Write out the conditions of perpendicularity:

$$\vec{n} \cdot \vec{u}_1 = 5a - c = 0, \qquad \vec{n} \cdot \vec{u}_2 = 2a + 2b = 0.$$

Since *any* perpendicular vector will do, we can choose one coordinate arbitrarily (but not 0), say $c = 5$ (to avoid fractions). Then $a = 1$ and $b = -1$:

$$\vec{n} = (1, -1, 5).$$

In other words, the plane is $x - y + 5z = 0$.

Method 2: The analogue of Theorem 1 for three dimensions is provided by the *vector cross product*:

$$\vec{x} \times \vec{y} \equiv \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \equiv \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}.$$

(The determinant notation here will be explained in Sec. 2.5, if you are not familiar with it.) The cross product of two vectors is always perpendicular to both of them. If $\vec{x}$ and $\vec{y}$ are not parallel to each other, then $\vec{x} \times \vec{y}$ is not zero. In our plane problem, therefore, $\vec{u}_1 \times \vec{u}_2$ is suitable for use as $\vec{n}$. We calculate

$$\vec{u}_1 \times \vec{u}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & 0 & -1 \\ 2 & 2 & 0 \end{vmatrix} = \begin{pmatrix} 2 \\ -2 \\ 10 \end{pmatrix}.$$

This is not the same vector $\vec{n}$ we got with the other method, but that is not a problem: The two vectors are proportional, and (3) can be multiplied by any constant (on both sides, of course!) without changing the plane.

Example 6. Find a parametric form of the plane passing through the points $(1, 0, 0)$, $(2, 2, 2)$, and $(-1, 0, 6)$.

SOLUTION: The difference between any two points on the plane is a vector parallel to the plane, therefore a suitable candidate for $\vec{u}_1$ or $\vec{u}_2$. We arbitrarily choose to subtract the first given vector from each of the other two:

$$\vec{u}_1 = (1, 2, 2), \qquad \vec{u}_2 = (-2, 0, 6).$$

Let us also choose the first vector as $\vec{x}_0$. Then

$$\vec{x} = s\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} s - 2t + 1 \\ 2s \\ 2s + 6t \end{pmatrix}.$$

Example 7. Find the equation form of the plane in the previous example.

SOLUTION: Reasoning as we did for lines, we see that the equation is $\vec{n} \cdot \vec{x} = d$, where $\vec{n}$ is perpendicular to the differences between any points in the plane (a clumsy vectorial way of saying "perpendicular to the plane"), and $d = \vec{n} \cdot \vec{x}_0$ for any point $\vec{x}_0$ in the plane. Therefore, it is easy to get these ingredients from the parametric form found in the previous example. By either of the methods in Example 5 we find that $\vec{n} = (6, -5, 2)$ is a vector perpendicular to $\vec{u}_1$ and $\vec{u}_2$. Using the $\vec{x}_0$ in Example 6, we get $\vec{n} \cdot \vec{x}_0 = 6$. Thus the equation of the plane is

$$6x - 5y + 2z = 6.$$

We summarize the principles used in solving these examples in the following theorem.

Theorem 2: In $\mathbf{R}^3$, the parametrized plane (4)

$$\vec{x} = s\vec{u}_1 + t\vec{u}_2 + \vec{x}_0$$

consists precisely of all the points satisfying the equation (3)

$$\vec{n} \cdot \vec{x} = d,$$

where $\vec{n}$ is any nonzero vector perpendicular to $\vec{u}_1$ and $\vec{u}_2$, and $d = \vec{n} \cdot \vec{x}_0$. (Here it is understood that $\vec{u}_1$ and $\vec{u}_2$ are not zero and are not parallel to each other.)

PROOF: We must show that the set of points (4) is the same as the set of points satisfying the equation (3). Therefore, we check that each set is a subset of the other one.

(i) If $\vec{x}$ is of the form (4), and $\vec{n}$ and d are as described, then taking the dot product of (4) with $\vec{n}$ indeed yields $\vec{n} \cdot \vec{x} = d$.

(ii) If $\vec{x}_1$ and $\vec{x}_2$ are two solutions of (3), then $\vec{x}_1 - \vec{x}_2$ is perpendicular to $\vec{n}$. (The proof of this is the same as that of the corresponding statement for lines, given earlier.) Since $\vec{x}_0$ solves (3), it follows that every solution $\vec{x}$ can be written as

$$\vec{x} = \vec{x}_0 + (\vec{x} - \vec{x}_0) = \vec{x}_0 + \vec{u},$$

where $\vec{u} \cdot \vec{n} = 0$. As in Example 4, it is easy to see that there will always be a two-parameter family of vectors $\vec{u}$ satisfying this perpendicularity condition. Thus the solution space of (3) can be represented in the form (4) for *some* $\vec{u}_1$ and $\vec{u}_2$. It is geometrically obvious that these can be chosen to be the same $\vec{u}_1$ and $\vec{u}_2$ that we started from; a formal proof of this requires some concepts that will be developed in later chapters.

We have shown that (1) and (2) are alternative descriptions of lines in two-dimensional space, and that (3) and (4) are alternative descriptions of planes in three-dimensional space. We observed that lines in $\mathbf{R}^3$ also have the parametrical representation (2). It is natural to ask whether a line in three dimensions has an equation form, analogous to (1) and (3). Since a line has only one free parameter, it is clear that there must be *two* conditions relating the three coordinates of a point on the line. In fact, since the intersection of two planes is a line (unless the planes are parallel), a line will be defined by *two* equations of the type (3). We leave further investigation of this situation to the exercises.

Exercises

1.2.1 Express in parametric form ($\vec{x} = t\vec{u} + \vec{x}_0$):

(a) The line through the origin in $\mathbf{R}^3$ parallel to the vector $(1, 0, 1)$.

(b) The line in $\mathbf{R}^2$ through the points $(1, 0)$ and $(0, -1)$.

(c) The line through $(2, 3)$ parallel to the vector $(1, 2)$.

(d) The line through the points $(1, 5, 7)$ and $(2, 10, 14)$.

1.2.2 Find an equation of the form $ax + by = c$ for these lines in $\mathbf{R}^2$.

(a) The line through the points $(4, 7)$ and $(2, -1)$.

(b) The line with parametric equation $\vec{x} = t(1, 1) + (4, -1)$.

(c) The line through the origin parallel to $(5, 1)$.

(d) The line with parametric equation $\vec{x} = t(0, 1) + (-2, -1)$.

1.2.3 Express in parametric form the plane defined by the equation $9x - 3y + z = 2$.

1.2.4 Let $\vec{n} = (-1, 0, 2)$. The equation $\vec{n} \cdot \vec{x} = 5$ defines a plane in $\mathbf{R}^3$. Express the plane in the parametrized form $\vec{x} = s\vec{u}_1 + t\vec{u}_2 + \vec{x}_0$.

1.2.5 Find an equation ($ax + by + cz = d$) for:

(a) The plane with parametric form $\vec{x} = s(1, 0, -1) + t(2, 1, 2)$.

(b) The plane through the origin perpendicular to $(2, 2, 1)$.

1.2.6 Find both a parametric form and an equation form for the plane passing through $(1, 0, 0)$, $(1, 1, 1)$, and the origin.

1.2.7 Express in equation form the plane whose parametric form is

$$\vec{x} = s(1, -1, 2) + t(2, 2, 2) + (5, 0, 2).$$

1.2.8 Find an equation and a parametric representation for the plane passing through the points $(1, 0, 1)$, $(2, 3, 1)$, $(5, 4, 5)$.

1.2.9 Find an equation and a parametric representation for the plane perpendicular to $\vec{n} = (3, 1, 1)$ and passing through $(4, 7, -1)$.

1.2.10 Prove Theorem 1 (including the statement in parentheses). HINT: Besides perpendicularity, what does "rotation through a right angle" entail?

1.2.11 Suppose that the line $\vec{x} = t\vec{u} + \vec{x}_0$ passes through the origin, but nevertheless $\vec{x}_0$ is not $\vec{0}$. How are $\vec{u}$ and $\vec{x}_0$ related in this situation?

1.2.12 Give an example of a plane $\vec{x} = s\vec{u}_1 + t\vec{u}_2 + \vec{x}_0$ such that (1) the plane passes through the origin, and (2) $\vec{x}_0$ is not $\vec{0}$ and is not parallel to either $\vec{u}_1$ or $\vec{u}_2$.

1.2.13 The analogue of Theorem 2 for a *line in* $\mathbf{R}^2$ was not stated formally as a theorem; you have to search through the expository text to find all the relevant conclusions. Summarize them formally, in analogy with Theorem 2.

1.2.14 In the notation of Theorem 2, show that if $\vec{n}$ has length 1, then $|d|$ is the distance of the plane from the origin.

1.2.15 The intersection of the planes

$$x + y + z = 2 \qquad \text{and} \qquad x - 2y + z = 0$$

is a line in $\mathbf{R}^3$. Find a parametric representation of this line. (*Suggestion:* Set the coordinate z equal to the parameter t.)

1.2.16 Characterize the line $\vec{x} = t(1, 0, -1)$ by a pair of equations, $\vec{n}_1 \cdot \vec{x} = 0 = \vec{n}_2 \cdot \vec{x}$. (Find two distinct planes containing the line. There are many different correct answers!)

1.2.17 Let $\vec{x} = (1, 0)$ and $\vec{y} = (2, 1)$. On a piece of graph paper plot and label the points $t\vec{x} + (1 - t)\vec{y}$ for $t = -0.5$, 0, 0.2, 0.5, 0.9, 1, and 1.2. From this example, formulate a general principle. (What is special about the points corresponding to $0 \leq t \leq 1$? These points are called *convex combinations* of the two given vectors.)