

Chapter 1
Vectors

1.1 Vectors that You Know

Vectors (and things made out of vectors or related to them) are the main subject matter of this book. Instead of starting with a precise mathematical definition of a vector, we give an informal, intuitive definition and discuss some examples.

> *Vectors are things that can be* **added to each other** *and* **multiplied by numbers**.

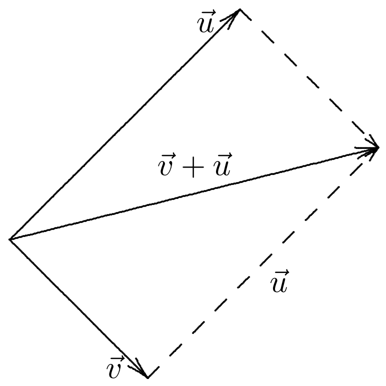
This definition assumes that we all have intuitive notions of addition and multiplication, which can be carried over from ordinary numbers to other kinds of objects. This will be seen to be true for each type of example we discuss. A later chapter contains the unambiguous, formal definitions of "addition", "multiplication", and "vector" that historically arose out of such examples.

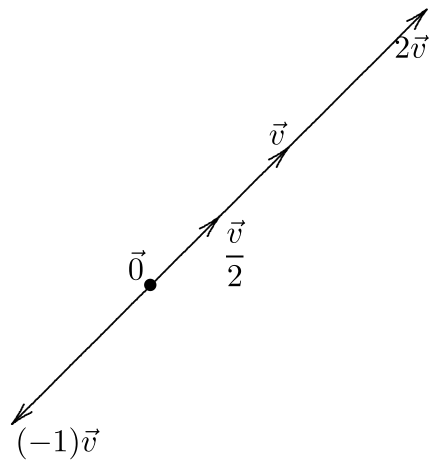
Example 1. Probably the most familiar vectors are those introduced in physics courses and characterized as *physical quantities with both magnitude and direction*. These include forces, velocities, electric fields, temperature gradients, magnetic moments. They are customarily drawn as arrows.

Within this geometrical conception of vectors as arrows, there is a well known construction for *adding* two vectors, shown in the drawing below. Initially the two arrows are thought of as rooted at the same point (the *origin*, or *zero vector*). One way of describing the sum is to slide one of the vectors (say $\vec{u}$) along the other one ($\vec{v}$), without rotating it or changing its length, so that the tail of $\vec{u}$ is at the head of $\vec{v}$. Then the sum $\vec{v}+\vec{u}$ is the arrow with its tail at the tail of $\vec{v}$ and its head at the head of $\vec{u}$, so that the three vectors form a triangle. Entirely equivalent to this "triangle rule" is the "parallelogram rule": Draw the parallelogram with $\vec{u}$ (in its original position) and $\vec{v}$ as adjacent sides. Then $\vec{v}+\vec{u}$ is the arrow pointing along the diagonal of the parallelogram starting from the origin.

The other half of the drawing shows how one *multiplies* a vector by a number geometrically: Simply put, the vector $\vec{v}$ is stretched by the numerical factor r. The word "stretching" is appropriate if r is a positive number greater than 1. If $0 \leq r < 1$, then "shrinking" is a better description. If r is negative, the arrow is reflected (so that it points in the opposite direction from the origin) in addition to a stretching or shrinking.

In this discussion we have tacitly assumed that r is a *real* number. Types of vectors that can be multiplied by *complex* numbers also arise. In this book we will deal mostly





with real vector spaces. When complex vector spaces come up, we will call special attention to them; otherwise, all numbers are assumed to be real.

Example 2. Vectors are encountered in elementary courses also in the form of *n-tuples of real numbers*. A 2-tuple is a *pair*, such as $(2, 3)$, and the vector space of all such pairs is called $\mathbf{R}^2$. Similarly, $\mathbf{R}^3$ consists of all strings of three numbers, and so on. Two pairs are added "componentwise", or "slot by slot":

$$(2,3) + \left(\tfrac{1}{2}, -1\right) = \left(\tfrac{5}{2}, 2\right).$$

More formally, we can write down the definition of the sum of two vectors in any $\mathbf{R}^n$:

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) \equiv (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

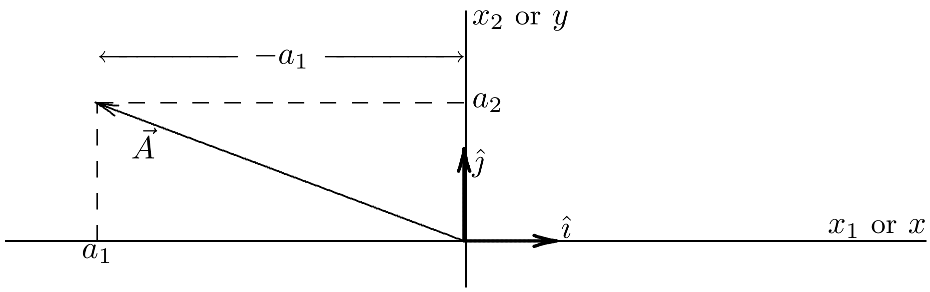
Similarly, multiplication is defined componentwise:

$$3(1,3,1) = (3,9,3);$$

$$r(a_1, a_2, \ldots, a_n) \equiv (ra_1, ra_2, \ldots, ra_n).$$

(The symbol "$\equiv$" means "equal by definition".)

Example 1 can be related to Example 2 by introducing a coordinate system into the geometrical space of Example 1:



Each physical vector is identified with a string of three numbers, (x_1, x_2, x_3) or (x, y, z). For example, the y component a_2 of the vector $\vec{A}$ is its projection onto the y axis, measured as a multiple of the *unit vector* or *basis vector* $\hat{\jmath}$ along that axis.

Indeed, we understand this sitation so well that we often think of Examples 1 and 2 as being the same thing! (Technically, one says that these two spaces of vectors are *isomorphic.*) Note, however, that the correspondence between them depends on the coordinate system. Introduce a rotated set of axes, and the same physical vector will correspond to a different string of numbers.

Important Notational Remark: There are several notations for vectors and their coordinates, and for basis vectors, which have grown up in connection with various applications. It is important to be able to deal with all of them and to tolerate occasional inconsistencies in notation. For example, a two-dimensional vector may be written in the various ways

$$(3,1) = 3\hat{\imath} + \hat{\jmath} = 3(1,0) + (0,1) = 3\hat{e}_1 + \hat{e}_2\,,$$

or, more generally,

$$\vec{A} = (a_1, a_2) = (A_1, A_2) = a_1\hat{e}_1 + a_2\hat{e}_2 = A_x\hat{\imath} + A_y\hat{\jmath},$$

etc. It is an unpleasant but unavoidable fact that sometimes numerical subscripts must be used to distinguish *different vectors* from each other, as well as, or instead of, to distinguish the *different coordinates* of the *same* vector. Since this issue will arise frequently in exercises and examples in the rest of this book, we shall belabor it a bit here. When we need to discuss two vectors in a two-dimensional space, we might call the vectors $\mathbf{x}$ and $\mathbf{y}$ and write them out as

$$\mathbf{x} = (x_1, x_2), \qquad \mathbf{y} = (y_1, y_2).$$

On another occasion, however, we may call the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ and write them out as

$$\mathbf{x}_1 = (x_1, y_1), \qquad \mathbf{x}_2 = (x_2, y_2).$$

Yes, this can be confusing, but an attempt to stick to a consistent notation would be misguided. Both notational conventions are used in the Real World, each has advantages under certain circumstances, and you must be prepared to handle whichever notation arises in any particular problem.

Remark on Terminology: Strictly speaking, a *coordinate* of the vector $(3,1) = 3\hat{\imath} + \hat{\jmath}$ is one of the *numbers*, 3 and 1, that appear in its expansion as a linear combination of basis vectors; whereas a *component* of the vector is another *vector*, the part of the given vector that points along one of the basis vectors: the component of $(3,1)$ along $\hat{\imath}$ is $3\hat{\imath}$ or $(3,0)$. However, "component" is frequently used also to mean the same thing as "coordinate"; indeed, "coordinate" sounds strange in nongeometrical contexts (such as the example in the next paragraph).

It should be noted that n-tuples also occur in contexts where the directions involved are not in physical space. For example, in an economic or business application the numerical components a_j , b_j , ... ($j = 1, \ldots, n$) of some vectors $\vec{a}$, $\vec{b}$, ... may be the prices, production quantities, and so on of n commodities. For instance, suppose a shirt factory produces per month

b_1 T-shirts of size S,

b_2 T-shirts of size M,

b_3 T-shirts of size L,

b_4 T-shirts of size XL,

and the company prices size-S shirts at p_1 dollars each, etc. Then the production level of the factory is summarized by the vector $\vec{b}$; the vector $3\vec{b}$ is what would be produced by three identical such factories; and

$$\sum_{j=1}^{4} p_j b_j \equiv \vec{p} \cdot \vec{b} \tag{1}$$

is the total revenue brought in by selling all the shirts produced.

In the expression (1) we recognize the familiar *vector dot product*, which also appears in such formulas from physics as $W = \vec{F} \cdot \vec{x}$ for the work done by a force that moves a body through a displacement $\vec{x}$. This operation of multiplying two vectors to get a number is not part of the definition of a general vector space; it is "extra structure" that exists in some concrete vector spaces but not others. It will be treated further in Chapter 6. (See also the discussions of row and column vectors in Sections 2.4, 3.2, and 4.2.)

To most students the word "vector" already calls to mind one or the other of the two types of vectors just discussed. The next two examples are equally familiar mathematical objects, but possibly you have never thought of them as vectors.

Example 3: *Polynomials.* Consider the power functions $\{1, t, t^2, t^3, \ldots\}$. These can be added together with numerical coefficients: e.g., $3t^3 - t + 2$. (Such a thing is called a *linear combination* of the vectors you start with.) We are used to manipulating these coefficients just like the components of ordinary vectors: We know how to add them by combining terms,

$$(t^2 + 3) + (t - 5) = t^2 + t - 2,$$

and we also know that the result can't be simplified any further. Note that the powers are playing the same role as the unit vectors $\{\hat{\imath}, \hat{\jmath}\}$ along the coordinate axes in Example 2. In each case we have a certain list of vectors from which all the vectors in the space can be built up by linear combination (and none of the vectors in the list can be left out); such a list is called a *basis*. The crucial difference is that in the vector space of polynomials the basis list is infinite (although any particular polynomial contains only finitely many terms).

Example 4: *Solutions of homogeneous linear differential equations.* Consider the ordinary differential equation

$$\frac{d^2y}{dt^2} + 4y = 0.$$

One of the first things taught in a course on differential equations is that this type of equation is most easily solved by means of complex numbers. The exponential of an imaginary number is defined as

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{2}$$

and the resulting exponential function of a complex variable can be shown to possess the familiar algebraic and calculus properties of the exponential of a real variable. Therefore,

$$y = f(t) = A\,e^{2it} + B\,e^{-2it}, \tag{3}$$

with A and B arbitrary complex numbers, is a solution of the differential equation. Furthermore, every complex-valued solution is of this form. The space of all these is a vector space. (We get new solutions by adding old ones together and by multiplying old ones by numbers. This is called the "principle of superposition" for homogeneous linear equations.) In fact, the formula sets up an identification between these vectors and the pairs of *complex* coefficients: $f \leftrightarrow (A, B)$. This is another example of an *isomorphism*, and the two exponential functions are another example of a basis.

A different isomorphism is given by the equally valid formula

$$f(t) = C\cos(2t) + D\sin(2t). \tag{4}$$

Formula (2) and the formulas

$$e^{-i\theta} = \cos\theta - i\sin\theta,$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{5}$$

that follow from it enable us to write C and D in terms of A and B or vice versa. (If this is not already familiar to you, you should do Exercise 1.1.12 now!) The passage from (3) to (4) is very much like the effect of a *rotation of axes* in Examples 1 and 2. If we are interested only in *real-valued* solutions, we'll prefer the trigonometric basis in (4) and require C and D to be real instead of complex. On the other hand, the exponential basis in (3) is very useful in finding the solutions in the first place. (Just substitute $y = e^{rt}$ and solve for r. The reason this trick works is that the derivative of an exponential function is a numerical multiple of the function itself. Such a function is an *eigenvector* of the operation of differentiation — see Chapter 8.)

Many of the facts about linear ordinary differential equations taught in differential-equation courses are actually general principles of linear algebra (vector-space theory), not special to ordinary differential equations. Understanding this makes it easy to see how to extend those methods to *partial* differential equations, for instance.

A central theme of this book — and of all mathematics — is the "up and back down" process demonstrated in the broadening of the vector concept from Examples 1 and 2 to include Examples 3 and 4. Experience with the comparatively elementary concepts of geometrical and numerical vectors leads to the formulation of a more abstract and general concept of a vector space (whose details we have postponed to Chapter 3). One then recognizes vectors in many other concrete situations, such as differential equations, and the abstract concept and the theorems derived from it provide powerful new tools for understanding those subjects and solving problems within them.

Exercises

1.1.1 Let $\vec{u} = (1, 2)$ and $\vec{v} = (1, -1)$. On graph paper, sketch $\vec{u}$ and $\vec{v}$ as arrows rooted at the origin. Then, from the geometrical definitions of the vector operations, sketch (and label)

(a) $\vec{u} + \vec{v}$,

(b) $\vec{u} - \vec{v}$,

(c) $3\vec{v}$,

(d) $\frac{1}{2}\vec{u} - 2\vec{v}$.

(A protractor, as well as a ruler, will be helpful.) Finally, check how well your results agree with the arithmetical definitions of these vectors. (Correct any gross errors in the sketch, but accept minor inaccuracies as the inevitable result of the approximate nature of the sketching process.)

1.1.2 Let $\vec{x} = (1, 2, 3)$, $\vec{y} = (1, -1, 1)$, $\vec{z} = (-2, 2, -2)$. Calculate

(a) $\vec{x} + \vec{y}$,

(b) $\vec{z} - \vec{x}$,

(c) $3\vec{x} - 4\vec{y} + \vec{z}$,

(d) $\vec{z} + 2\vec{y}$.

1.1.3 For the vectors defined in Exercise 2, calculate

(a) $\vec{x} \cdot \vec{y}$,

(b) $\vec{x} \cdot \vec{z}$,

(c) $\vec{z} \cdot \vec{y}$.

1.1.4 Compare parts (a) and (b) of Exercise 3, and formulate the general theorem that the comparison illustrates.

1.1.5 For the vectors defined in Exercise 2, write out and simplify the formula $\vec{v} \equiv a\vec{x} + b\vec{y} + c\vec{z}$, where a, b, and c are arbitrary numbers. Then show that:

(a) Any change in b can be compensated by a change in c (so that $\vec{v}$ is unchanged).

(b) The coefficient a cannot be changed without changing $\vec{v}$, regardless of what happens to the other two coefficients.

1.1.6 Simplify:

(a) $3(t^2 + 3t + 2) - 10(t^3 + t^2 - 10) + 4(t-1)^2 - t + 5$

(b) $-6(\cos(2t) + 3\sin(2t)) + 5\cos(2t) + 3e^{2it}$

1.1.7 Let $\vec{x} = (1,1)$ and $\vec{y} = (0,2)$.

(a) *(trivial)* Express $\vec{x}$ as a linear combination of the basis vectors $\hat{\imath}$ and $\hat{\jmath}$.

(b) Express $\hat{\imath}$ as a linear combination of $\vec{x}$ and $\vec{y}$.

1.1.8 (a) Simplify $(t-3)^3 + 5(t-3)^2 - 10(t-3) + 1$ into the standard form for polynomials (a linear combination of powers of t).

(b) Express $t^2 + 2$ as a linear combination of the powers of $(t-3)$.

1.1.9 (a) Express $3\cos t - 2\sin t$ as a linear combination of e^{it} and e^{-it}.

(b) Express $e^{3it} - 2e^{-3it}$ as a linear combination of the trigonometric functions $\sin(3t)$ and $\cos(3t)$.

1.1.10 Let $\vec{u} = (10, 5, 0.1)$ represent the amount of flour, sugar, and baking powder required to produce a dozen bagels, and let $\vec{v} = (20, 7, 0.5)$ be the corresponding vector for a loaf of bread. Provide in words an interpretation for $60\vec{u} + 200\vec{v}$.

1.1.11 Whizbang Supersystems Inc. manufactures three models of computer, the Nerdstation 1000, 3000, and 5000. Each is priced according to its name (the cheapest costs \$1000, etc.)

(a) Show how the total revenue of the company can be expressed as a dot product of two vectors. (Define notation clearly.)

(b) The fixed cost of operating the Whizbang factory is \$100,000 per year. The production costs of Nerdstations is expressed by the vector $\vec{c} = (500, 1000, 1500)$. (That is, it costs \$500 to make one Model 1000, etc.) Find a formula in vector notation for the company's annual profit.

1.1.12 (a) Prove (5) from (2) (assuming the well known facts $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$).

(b) Using (2) and (5), work out the equations expressing A and B in (3) in terms of C and D in (4), and the equations expressing C and D in terms of A and B.