Hyperbolic Functions and the Twin Paradox

The basic hyperbolic functions are defined as

$$\cosh x \equiv \tfrac{1}{2}(e^x + e^{-x}), \qquad \sinh \equiv \tfrac{1}{2}(e^x - e^{-x}).$$

The notation ch, sh is also used (especially in European books). Their main usefulness is that every solution of

$$\frac{d^2 y}{dx^2} = k^2 y$$

can be written

$$y(x) = C \cosh(kx) + D \sinh(kx),$$

and the constants are neatly related to the initial data:

$$C = y(0), \quad D = \frac{y'(0)}{k}.$$

The hyperbolic functions satisfy a long list of identities closely parallel to well known identities for trigonometric functions. Some of these are:

$$\frac{d}{dx} \sinh x = \cosh x, \qquad \frac{d}{dx} \cosh x = \sinh x,$$

$$\cosh^2 x - \sinh^2 x = 1,$$

$$\cosh(-x) = \cosh x, \qquad \sinh(-x) = -\sinh x,$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y,$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y,$$

$$\sinh x = x + \frac{x^3}{6} + \cdots, \qquad \cosh x = 1 + \frac{x^2}{2} + \cdots$$

(that is, the power series for sin and cos with all the minus signs removed). All of these identities are easily derived or checked from the definitions and the elementary properties of the exponential function.

On the other hand, the qualitative behavior of these functions is very different from the trig functions: $\cosh x \geq 1$ while $\sinh x$ takes on all values from $-\infty$ to ∞. They are decidedly *not* periodic. As $x \to \pm\infty$, one or the other of the exponential terms dominates, so the hyperbolic function grows like $\frac{1}{2}e^{|x|}$. The function

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

approaches ± 1 as $x \to \pm\infty$, and its graph resembles that of the *inverse tangent* function.

[With luck, graphs will be inserted here someday.]

WHY ARE THEY CALLED HYPERBOLIC?

Hyperbolic functions have the same relation to a right-angled hyperbola as trigonometric functions have to a circle. (In fact, trig functions are sometimes called "circular functions".)

[Again, printed versions of the graphs must wait.]

First, recall the basic diagram relating the trig functions to a right triangle (with side lengths $\cos u$, $\sin u$, and 1) and the corresponding sector of a circle. Here u has three different geometrical interpretations:

1. the angle POQ opposite $\sin u$ and adjacent to $\cos u$.

2. the length of the arc AP opposite that angle.

3. twice the area of the sector AOP bounded by that arc and the line segments OA and OP.

The equation of the circle is $x^2 + y^2 = 1$ (reflecting the trig identity $\cos^2 u + \sin^2 u = 1$), and the circle is traced out parametrically as $0 \leq u < 2\pi$.

Now consider the right-hand branch of the hyperbola with equation $x^2 - y^2 = 1$, which will reflect the hyperbolic identity $\cosh^2 u - \sinh^2 u = 1$. In fact, that branch will be traced out parametrically as $-\infty < u < \infty$. Here u is *not* an angle (at least, not in the usual sense). However, there are close parallels to the second and third statements about the circle.

First, it can be shown that u equals twice the area of the "shark's fin" bounded by OP, OA, and the arc AP.

PROPER TIME

Second, although u is not the length of the arc AP, there is a way in which it is analogous to an arc length. Recall that ordinary arc length is defined by the schematic formula

$$ds = \sqrt{dx^2 + dy^2}$$

— which means that for any parametrized curve $x = x(u)$, $y = y(u)$, the length of a segment is

$$s = \int_{u_1}^{u_2} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du.$$

Thus for the circle we have

$$\text{arc}\, AP = \int_0^u \sqrt{\sin^2 \tilde{u} + \cos^2 \tilde{u}}\, d\tilde{u} = u,$$

which was assertion 2 about the circle. In the hyperbolic case the corresponding integral for $\text{arc}\, AP$ is not equal to u. However, if we define a new quantity by

$$d\sigma = \sqrt{|dy^2 - dx^2|},$$

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$$\sigma = \int_{u_1}^{u_2} \sqrt{\left| \left(\frac{du}{dy} \right)^2 - \left(\frac{dx}{du} \right)^2 \right|} \, du,$$

then for the hyperbola the σ of the curve segment AP is

$$\sigma = \int_0^u \sqrt{\cosh^2 \tilde{u} - \sinh^2 \tilde{u}} \, d\tilde{u} = \int_0^u d\tilde{u} = u.$$

This definition actually has a physical application. Let y stand for *time*, and consider motion in a straight line, so we can ignore two of the three space coordinates. Then a curve on our diagram is a graph of the position, x, of an object as a function of time — provided that the curve crosses each horizontal line only once. (It may seem strange that we are marking the independent variable of the function on the vertical axis, but that is the tradition in relativistic physics.) Furthermore, the slope of the curve must be greater than 1 to enforce the condition that the object moves more slowly than light (which has speed 1 in our units). Now, the basic physical law involving σ is

> *The time measured by any (accurate) clock whose motion is described by such a curve is the parameter σ of that curve.*

Therefore, σ is called *proper time*.

THE TWIN "PARADOX"

A famous prediction of Einstein's theory of special relativity: Consider two twins. One stays at home. The other takes a trip on a fast rocket ship and returns home. Then he will be *younger* than his sibling who stayed at home.

Let the moving twin's round trip be described by the hyperbolic segment $P'AP$, the stationary twin's (trivial) trip by line segment $P'QP$. The equation of that line is $x = \cosh u$ (a constant function of y). We know that the proper time σ of $P'AP$ equals $2u$, since we calculated the contribution of the top half to be u. What is the proper time of $P'QP$?

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$$\sigma = \int \sqrt{|dy^2 - dx^2|} = \int_{-y}^{y} \sqrt{1 - \left(\frac{dx}{d\tilde{y}}\right)^2} \, d\tilde{y}$$

$$= \int_{-y}^{y} d\tilde{y} = 2y = 2\sinh u.$$

That is,

age of stationary twin $= 2\sinh u > 2u =$ age of moving twin.

There is nothing paradoxical about this result, because the two twins are not equivalent. One is accelerated (moving on a curved path in space-time), while the other is moving at a constant speed (moving on a straight path in space-time). (In the natural coordinate system for the problem, that speed is 0 and the straight path is vertical, but those features are irrelevant to the main point.) The conclusion should be no more surprising than the fact that the arc length of a semicircle is larger than the diameter $(\pi r > 2r)$.