Math. 312 17 November 2000

Test C – Solutions

1. *(45 pts.)* Solve the heat equation in a disk with insulated edge:

$$u = u(t, r, \theta), \qquad t > 0, \quad 0 \le r < 3, \quad 0 \le \theta < 2\pi,$$

$$\frac{\partial u}{\partial t} = \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

$$\frac{\partial u}{\partial r}(t, 3, \theta) = 0,$$

$$u(0, r, \theta) = f(r, \theta).$$

(Be as explicit as you can about the eigenvalues and eigenfunctions that arise. Sketching one of the eigenfunctions would be a good idea.)

In a word, we expect Bessel functions. But let's go through the steps of variable separation to get there.

Consider $u_{\text{sep}}(t, r, \theta) = T(t)R(r)\Theta(\tau)$. Then

$$T'R\Theta = TR''\theta + \frac{1}{r}TR'\Theta + \frac{1}{r^2}TR\Theta''.$$

Thus

$$\frac{T'}{T} = -\omega^2 = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}.$$

(We should come back later and check that the separation constant is always negative, but for a heat equation we expect most of the modes, at least, to decay in time.) Thus $T(t) = e^{-\omega^2 t}$.

Multiply the remaining equation by r^2 to separate the variables, and introduce another separation constant:

$$-\frac{\Theta''}{\Theta} = n^2 = r^2\frac{R''}{R} + r\frac{R'}{R} + \omega^2 r^2.$$

The periodic boundary conditions on Θ ensure that n^2 is really positive, and in fact that n is an integer. We have $\Theta(\theta) = e^{i\nu\theta}$, $\nu = \pm n$.

Now we're left with

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \left(\omega^2 - \frac{n^2}{r^2}\right) = 0,$$

a form of Bessel's equation. Let $z = \omega r$ and $Z(z) = R(r)$. Then $d/dr = \omega\, d/dz$, so

$$Z'' + \frac{1}{z}Z' + \left(1 - \frac{n^2}{z^2}\right)Z = 0$$

(where the primes now indicate differentiation with respect to z). The solutions of this are $J_n(z)$ and $Y_n(z)$, but because our region includes the origin and the solution must be smooth there, only J functions can appear. Thus $R(r) = J_n(\omega r)$. The boundary condition requires that $R'(3) = 0$. Therefore, 3ω must be a zero of the derivative of J_n. So, define y_{nj} to be the jth zero: $J'(y_{nj}) = 0$. Then $\omega_{nj} = \frac{1}{3}y_{nj}$.

Now, could ω^2 possibly be negative or 0? Extra credit to anybody who realized the following: For $n \neq 0$, the first zero of J_n' is smaller than the first zero of J_n , so our first eigenfunction has no nodes and there is no way that there could be another eigenfunction with an even smaller eigenvalue. (Alternative argument: The eigenfunction would have to be r^n (for $\omega = 0$) or the modified Bessel function $I_n(\kappa r)$ (for $\omega^2 = -\kappa^2 < 0$), and the derivatives of these functions have no zeros.) But for $n = 0$ the first zero of the derivative (other than the one at the origin) comes *after* the first derivative of J_0 itself, so there probably is another eigenfunction with no nodes lurking somewhere. In fact, the eigenvalue ω^2 is 0, and the eigenfunction is the constant function 1 (leading to $u_{\text{sep}}(t, r, \theta) = 1 \times 1 \times 1 = 1$, and hence to the term A_{00} in the next equation).

We are now ready to superpose the normal mode solutions u_{sep} ; this requires summing over n and j.

$$u(t, r, \theta) = A_{00} + \sum_{\nu=-\infty}^{\infty} \sum_{j=1}^{\infty} A_{\nu j} e^{-\omega_{nj}^2 t} J_n(\omega_{nj} r) e^{i\nu\theta},$$

where $n = |\nu|$ and $\omega_{nj} = \frac{1}{3} y_{nj}$.

Finally we have to determine the coefficients from the initial data.

$$f(r, \theta) = A_{00} + \sum_{\nu=-\infty}^{\infty} \sum_{j=1}^{\infty} A_{\nu j} J_n(\omega_{nj} r) e^{i\nu\theta}.$$

We can solve this either in one step, by taking inner products with the two-dimensional eigenfunctions $\phi_{\nu j} = e^{i\nu\theta} J_n(\omega_{nj} r)$, or in two steps, doing an ordinary Fourier series calculation and then a Fourier-Bessel series calculation. Thinking in the first way, one sees that

$$A_{00} = \frac{1}{9\pi} \int_0^{2\pi} d\theta \int_0^3 r \, dr \, f(r, \theta)$$

(the 9π is the area of the disk, which is the square of the norm of the eigenfunction 1). Let's get the other constants the other way:

$$\phi_n(r) \equiv \frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu\theta} f(r, \theta) \, d\theta = \sum_{j=1}^{\infty} A_{\nu j} J_n(\omega_{nj} r),$$

and then

$$A_{\nu j} = \frac{\int_0^3 f_\nu(r) J_n(\omega_{nj} r) \, r \, dr}{\int_0^3 J_n(\omega_{nj} r)^2 \, r \, dr} .$$

Whew!

2. *(45 pts.)* Let's study the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

on the interval $0 < x < 1$ with the boundary conditions

$$u(t, 0) = 0, \qquad \frac{\partial u}{\partial x}(t, 1) + \beta u(t, 1) = 0,$$

where β is a **positive** constant.

(a) Separate variables, find the eigenfunctions, and indicate graphically how to find the eigenvalues. Give an approximate formula for the large eigenvalues.

Let $u_{\rm sep}(t,x) = T(t)X(x)$. Then

$$\frac{T''}{T} = -\omega^2 = \frac{X''}{X},$$

and X must satisfy $X'(1)+\beta X(1) = 0$ and $X(0) = 0$. Thus $X(x) = \sin(\omega x)$ and $\omega\cos\omega + \beta\sin\omega = 0$. The eigenvalue equation is most conveniently written

$$\tan\omega = -\frac{\omega}{\beta},$$

which can be graphed exactly as in the class notes. For large n the nth solution is slightly greater than $(n-\frac{1}{2})\pi$. The positivity of β is the condition that excludes negative and zero solutions of ω^2.

(b) Solve the wave equation with initial data

$$u(0,x) = f(x), \qquad \frac{\partial u}{\partial t}(0,x) = g(x).$$

Superpose the normal-mode solutions found above, noting that there are two solutions to the time equation:

$$u(t,x) = \sum_{n=1}^{\infty}\bigl[a_n\sin(\omega_n x)\cos(\omega_n t) + b_n\sin(\omega_n x)\sin(\omega_n t)\bigr].$$

Then

$$f(x) = \sum_{n=1}^{\infty} a_n\sin(\omega_n x), \qquad g(x) = \sum_{n=1}^{\infty} b_n\omega_n\sin(\omega_n x).$$

Therefore,

$$a_n = \frac{\int_0^1 f(x)\sin(\omega_n x)\,dx}{\int_0^1 \sin(\omega_n x)^2\,dx}, \qquad b_n = \frac{\int_0^1 g(x)\sin(\omega_n x)\,dx}{\omega_n\int_0^1 \sin(\omega_n x)^2\,dx}.$$

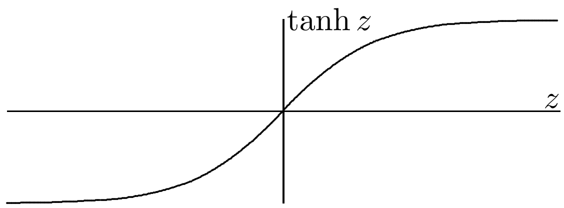
3. *(10 pts.)* Answer **ONE** of these. (Extra credit for both.) Relatively brief and qualitative answers are expected, not complete calculations.

(A) What would happen in Question 1 if the disk were replaced by an annulus (ring) with inner boundary $r = 1$? Suppose that that edge is held at a constant temperature, $u(t,1,\theta) = T$.

First of all, if $T \neq 0$ we need to subtract off a steady-state solution. It is easy to see that $v(t,r,\theta) = T$ is that solution, since it satisfies the heat equation and the other boundary condition.

Let $w = u - v$. Then w satisfies the same problem except that $u(t,1,\theta) = 0$ and $u(0,r,\theta) = f(r,\theta) - T \equiv g(r,\theta)$. Now we could separate variables in the same way as before, but this time the radial function $R(r) = Z(\omega r)$ would have to satisfy $R(1) = 0$ as well as $R'(3) = 0$. Since the origin is no longer in the region, there is no reason why Y_n can't appear. So $Z(z) = \alpha_n(\omega)J_n(z) + \beta_n(\omega)Y_n(z)$, and the equations $Z'(3\omega) = 0$ and $Z(\omega) = 0$ determine both the ratio of α to β and the values of ω that can occur.

(B) What would happen in Question 2 if the constant β were negative? *Useful information:* The *hyperbolic* tangent function has a graph like this, with slope 1 at the origin and asymptote 1 at $+\infty$:



First, in the graphical solution the graph of $-\frac{\omega}{\beta}$ now slopes up instead of down, so the nth solution is slightly less than $(n + \frac{1}{2})\pi$. (Actually, if $\beta < 1$ this straight line also intersects the first branch of the tangent function, so there is a zeroth solution somewhere less than $\frac{\pi}{2}$.)

Second, it is now possible to have a negative solution for ω^2. In that case, setting $\kappa^2 = -\omega^2$, we have $X(x) = \sinh(\kappa x)$ and

$$\tanh \kappa = \frac{\kappa}{\beta}\,.$$

If $\beta > 1$, this equation has one solution, as you can see by adding the diagonal straight line to the graph above. (The accompanying function of t will also be a sinh or cosh, so this mode is an *instability* in the system — probably making this problem physically implausible!) If $\beta < 1$ the negative eigenvalue does not exist, being replaced by the extra positive eigenvalue mentioned previously. If $\beta = 1$, zero is an eigenvalue with eigenfunction $X(x) = x$.