Math. 412 (Fulling) 29 September 2005

Test A – Solutions

Calculators may be used for simple arithmetic operations only!

1. *(15 pts.)* Classify each equation as linear homogeneous, linear nonhomogeneous, or nonlinear.

   (a) $\dfrac{\partial^2 u}{\partial t^2} + \left(\dfrac{\partial u}{\partial x}\right)^2 = x^2\cos(2x)$

nonlinear

   (b) $\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + 16u = 0$

linear homogeneous

   (c) $\dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial x^2} - x(1-x) = 0$

linear nonhomogeneous

2. *(35 pts.)* Let $f(x) = x$ for $-\pi \le x < \pi$.

   (a) Find the ("full") Fourier series for f (with $[-\pi, \pi]$ as the basic interval).

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

If $n \neq 0$,

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} x\,dx \\
&= \frac{1}{2\pi}\,\frac{1}{-in}\left[e^{-inx}x\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^{-inx}\,dx \right] \\
&= \frac{i}{2\pi n}(\pi e^{-in\pi} + \pi e^{in\pi} - 0) \\
&= \frac{1}{2\pi n}\,2\pi\cos(n\pi) = \frac{i(-1)^n}{n}\,.
\end{aligned}$$

If $n = 0$,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0.$$

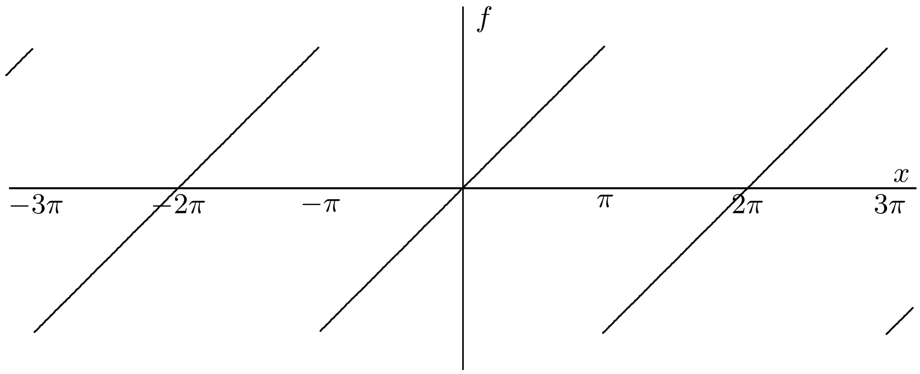
Thus

$$f(x) = \sum_{0\neq n=-\infty}^{\infty} \frac{i(-1)^n}{n}\, e^{inx}.$$

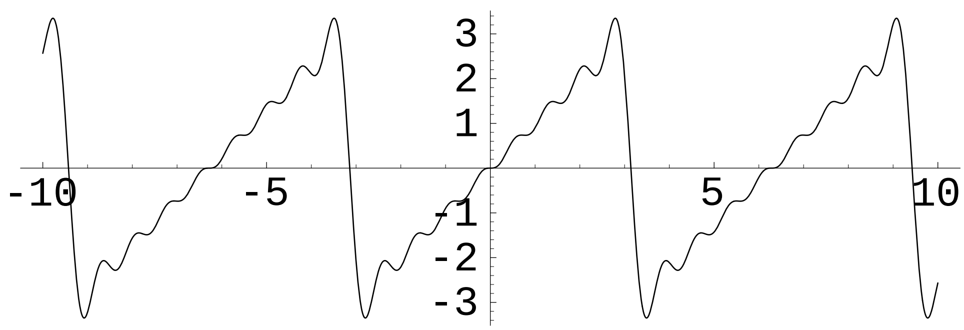
This can also be written

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n}\,\sin(nx).$$

(b) Over the interval $[-10, 10]$, sketch the function to which the series converges.



(c) Sketch a typical partial sum of the series (say the one with $|n| \le 8$). The sketch is not expected to be precise, just qualitative.



(d) Does the series converge

(i) uniformly?

No. The periodic extension is discontinuous at $x = N\pi$ (N odd).

(ii) pointwise?

Yes. The periodic extension is piecewise smooth.

(iii) in the mean?

Yes. The function is bounded and therefore obviously square-integrable over the finite interval. Alternatively, since $|c_n|^2 \propto n^{-2}$, the Parseval sum converges.

3. *(40 pts.)* Consider the wave equation on an interval,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi, \quad -\infty < t < \infty),$$

with boundary conditions

$$u(0,t) = 0 = u(\pi, t)$$

and initial conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

(a) Describe in words and sketches (and possibly a few equations) what the solution is like, assuming that $f(x)$ is a sharply peaked function such as $f(x) = e^{-10(x-1)^2}$.

The pulse will split into two halves, which move to left and right without changing shape. When a pulse hits an end of the interval, it reflects upside down. (For sketches see the similar problem in the Test A Solutions for Fall 2000. In that case the boundary condition was different, so the pulses did not invert upon reflection.) The d'Alembert formula for the solution is

$$u(x,t) = \tfrac{1}{2}[f(x-t) + f(x+t)],$$

where f is the odd periodic extension of the original f.

(b) Now assume instead that $f(x) = \sin x - \frac{1}{9}\sin(3x)$ (which is *not* very sharply peaked). Find a formula for $u(x,t)$ as a finite Fourier series. (There are several ways to do this, some quicker than others. *Think* before you launch a massive calculation.)

Method 1: Since f is already odd and 2π-periodic, we can use the d'Alembert formula immediately:

$$u(x,t) = \tfrac{1}{2}[\sin(x-t) + \sin(x+t)] - \tfrac{1}{18}[\sin(3(x-t)) + \sin(3(x+t))].$$

Now use trig identities to rearrange this into the Fourier form,

$$u(x,t) = \sin x \cos t - \tfrac{1}{9}\sin(3x)\cos(3t).$$

Method 2: Each term in f has the form appropriate to a normal mode for the wave equation on $[0,\pi]$ with Dirichlet boundary conditions, $\sin(nx)[a\cos(nt)v + b\sin(nt)]$. Since the initial time derivative is zero, the cosine terms don't appear. Thus

$$u(x,t) = \sin x \cos t - \tfrac{1}{9}\sin(3x)\cos(3t).$$

Method 3: Go through the whole process of separation of variables. (But I hope you didn't.)

4. *(35 pts.)* Solve Laplace's equation in a square,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < L, \quad 0 < y < L,$$

with the boundary conditions

$$u(0,y) = T, \quad u(L,y) = 0, \quad \frac{\partial u}{\partial y}(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,L) = f(x).$$

Note that $f(x)$ is an arbitrary function and T is a (nonzero) *constant*.

There are two nonhomogeneous conditions (affecting different variables), so we should construct the solution as a sum of two terms.

Best method: Since T does not depend on y, we can find a solution of the PDE and the x-boundary conditions that is independent of y. (This is like a steady-state solution, although y is not a time coordinate.) We have

$$V''(x) = 0, \quad V(0) = T, \quad V(L) = 0,$$

hence

$$V(x) = Ax + B, \quad B = T, \quad AL + B = 0,$$

so

$$V(x) = \frac{T}{L}\,(L - x).$$

Now let $w = u - V$. It must satisfy

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, \quad w(0,y) = 0, \quad w(L,y) = 0, \quad \frac{\partial w}{\partial y}(x,0) = 0, \quad \frac{\partial w}{\partial y}(x,L) = f(x).$$

(We don't need to subtract V from f , because f is the y *derivative* in this problem.) Standard separation of variables, $w_{\mathrm{sep}}(x,y) = X(x)Y(y)$, leads to

$$X''(x) + k^2 X(x) = 0, \quad X(0) = 0 = X(L), \quad Y''(y) - k^2 Y(y) = 0, \quad Y'(0) = 0.$$

As usual,

$$k = k_n \equiv \frac{n\pi}{L}\,, \quad X_n(x) = \sin(k_n x).$$

We can take $Y_n(y) = \cosh(k_n y)$.

Superposing, we get the general solution

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin(k_n x)\cosh(k_n y).$$

This is required to satisfy the remaining boundary condition,

$$f(x) = \sum_{n=1}^{\infty} b_n k_n \sin(k_n x)\sinh(k_n y).$$

Therefore,

$$b_n k_n \sinh(k_n y) = \frac{2}{L}\int_0^L f(x)\sin(k_n x)\,dx$$

determines b_n . Finally, $u(x,t) = V(x) + w(x,t)$.

Alternative method: Write $u(x,t) = v(x,t) + w(x,t)$, where

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \quad v(0,y) = T, \quad v(L,y) = 0, \quad \frac{\partial v}{\partial y}(x,0) = 0, \quad \frac{\partial v}{\partial y}(x,L) = 0,$$

and w satisfies the complementary equations, which turn out to be the same as in the other method. Separation of variables in the v problem yields

$$X''(x) - k^2 X(x) = 0, \quad X(L) = 0, \quad Y''(y) + k^2 Y(y) = 0, \quad Y'(0) = 0 = Y'(L).$$

Thus

$$k = k_n \equiv \frac{n\pi}{L}\,, \quad Y_n(y) = \cos(k_n y), \quad X_n(x) \propto \begin{cases} \sinh[k_n(L-x)] & \text{if } n \neq 0, \\ L - x & \text{if } n = 0. \end{cases}$$

The superposition is

$$v(x,y) = a_0(L-x) + \sum_{n=1}^{\infty} a_n \cos(k_n y) \sinh[k_n(L-x)].$$

It must satisfy

$$T = a_0 L + \sum_{n=1}^{\infty} a_n \cos(k_n y) \sinh(k_n L).$$

Thus if $n \neq 0$

$$a_n \sinh(k_n L) = \frac{2}{L} \int_0^L T \cos(k_n y)\, dy = 0.$$

For $n = 0$ we have

$$a_0 L = \frac{1}{L} \int_0^L T\, dy = T,$$

so

$$v(x,y) = \frac{T}{L}\,(L-x).$$