COMPOSITION OF LORENTZ TRANSFORMATIONS

$$U := \begin{bmatrix} 2 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

Find the Lorentz transformation into the rest frame of the observer with 4-velocity U. (It is understood that U_z = 0 and we ignore the z components of all vectors.)

$$-U_0^2 + U_1^2 + U_2^2 = -1$$

so this is indeed a normalized 4-velocity.

Method 1: Find the Lorentz transformation that eliminates the y component; then the one that eliminates the x component.

$$vy := \frac{U_2}{U_0} \qquad vy = 0.5 \qquad gamy := \frac{1}{\sqrt{1 - vy^2}} \qquad gamy = 1.155$$

$$Ly := \begin{bmatrix} gamy & 0 & -vy \cdot gamy \\ 0 & 1 & 0 \\ -vy \cdot gamy & 0 & gamy \end{bmatrix} \qquad Ly \cdot U = \begin{bmatrix} 1.732 \\ 1.414 \\ 0 \end{bmatrix} \qquad Uy := Ly \cdot U$$

$$vx := \frac{Uy_1}{Uy_0} \qquad vx = 0.816 \qquad gamx := \frac{1}{\sqrt{1 - vx^2}} \qquad gamx = 1.732$$

$$Lx := \begin{bmatrix} gamx & -vx \cdot gamx & 0 \\ -vx \cdot gamx & gamx & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Lx \cdot Uy = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

as desired.

$$L1 := Lx \cdot Ly \qquad L1 = \begin{bmatrix} 2 & -1.414 & -1 \\ -1.633 & 1.732 & 0.816 \\ -0.577 & 0 & 1.155 \end{bmatrix} \qquad L1 \cdot U = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Method 2: Same as before, but treat x first.

$$vx := \frac{U_1}{U_0} \qquad vx = 0.707 \qquad gamx := \frac{1}{\sqrt{1 - vx^2}} \qquad gamx = 1.414$$

$$Lx := \begin{bmatrix} gamx & -vx \cdot gamx & 0 \\ -vx \cdot gamx & gamx & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Lx \cdot U = \begin{bmatrix} 1.414 \\ 0 \\ 1 \end{bmatrix} \qquad Ux := Lx \cdot U$$

$$vy := \frac{Ux_2}{Ux_0} \qquad vy = 0.707 \qquad gamy := \frac{1}{\sqrt{1 - vy^2}} \qquad gamy = 1.414$$

$$Ly := \begin{bmatrix} gamy & 0 & -vy \cdot gamy \\ 0 & 1 & 0 \\ -vy \cdot gamy & 0 & gamy \end{bmatrix} \qquad Ly \cdot Ux = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{as desired.}$$

$$L2 := Ly \cdot Lx \qquad L2 = \begin{bmatrix} 2 & -1.414 & 9 \;\; -1 \\ -1 & 1.414 & 0 \\ -1.414 & 1 & 1.414 \end{bmatrix} \qquad L2 \cdot U = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Note that L1 and L2 are NOT equal, but both seem to transform into the rest frame of U.

$$L1 \cdot L2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.816 & 0.577 \\ 0 & -0.577 & 0.816 \end{bmatrix}$$

The difference between them is a spatial rotation! This is an inherent ambiguity in "the rest frame of U".

Method 3: Choose spatial axes so that the velocity is in the x direction. Solve that 1-dimensional problem, then rotate back.

$$tantheta := \frac{U_2}{U_1} \qquad costheta := \frac{1}{\sqrt{1 + tantheta^2}}$$

$$sintheta := tantheta \cdot costheta \qquad sintheta^2 + costheta^2 = 1$$

$$sintheta = 0.577 \qquad costheta = 0.816$$

$$R := \begin{bmatrix} 1 & 0 & 0 \\ 0 & costheta & sintheta \\ 0 & -sintheta & costheta \end{bmatrix} \qquad Ur := R \cdot U \qquad Ur = \begin{bmatrix} 2 \\ 1.732 \\ 0 \end{bmatrix}$$

$$vx := \frac{Ur_1}{Ur_0} \qquad vx = 0.866 \qquad gamx := \frac{1}{\sqrt{1 - vx^2}} \qquad gamx = 2$$

$$Lr := \begin{bmatrix} gamx & -vx \cdot gamx & 0 \\ -vx \cdot gamx & gamx & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Lr \cdot Ur = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$L3 := R^{-1} \cdot Lr \cdot R$

$$L3 = \begin{bmatrix} 2 & -1.414 & -1 \\ -1.414 & 1.667 & 0.471 \\ -1 & 0.471 & 1.333 \end{bmatrix} \qquad L3 \cdot U = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$L3 \cdot L1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.962 & -0.272 \\ 0 & 0.272 & 0.962 \end{bmatrix} \qquad L3 \cdot L2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.943 & 0.333 \\ 0 & -0.333 & 0.943 \end{bmatrix}$$

Thus L3 leads to a frame with a spatial orientation intermediate between those of L1 and L2.

Method 4: Let's construct L3 by a more geometrical method.

$$Uspat := \begin{bmatrix} 0 \\ U_1 \\ U_2 \end{bmatrix} \qquad N := \frac{Uspat}{\sqrt{{U_1}^2 + {U_2}^2}} \qquad N = \begin{bmatrix} 0 \\ 0.816 \\ 0.577 \end{bmatrix}$$

N is the unit vector in the direction of motion. The Lorentz transformation must leave alone spatial components perpendicular to N, while it mixes the spatial component parallel to N with the time component.

$P(V) := N \cdot (N \cdot V)$

$$P(U) = \begin{bmatrix} 0 \\ 1.414 \\ 1 \end{bmatrix}$$

P(V) is the component of V parallel to N.

$$Pr := \begin{bmatrix} N_0 \cdot N_0 & N_0 \cdot N_1 & N_0 \cdot N_2 \\ N_1 \cdot N_0 & N_1 \cdot N_1 & N_1 \cdot N_2 \\ N_2 \cdot N_0 & N_2 \cdot N_1 & N_2 \cdot N_2 \end{bmatrix} \qquad Pr = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.667 & 0.471 \\ 0 & 0.471 & 0.333 \end{bmatrix}$$

Pr is the matrix of the linear map V -> P(V). Check this:

$$P\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad P\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0.667 \\ 0.471 \end{bmatrix} \qquad P\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0.471 \\ 0.333 \end{bmatrix}$$

$$identity(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Pr \cdot identity(3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.667 & 0.471 \\ 0 & 0.471 & 0.333 \end{bmatrix}$$

Now the desired Lorentz transformation, acting on V, must give
time component gamma V_0 - gamma beta (N*V),
parallel component gamma P(V) - gamma beta V_0 N,
and perpendicular component V_spatial - P(V).
See Jackson, Classical Electrodynamics, Sec. 11.2, for these equations in better notation than the computer is capable of.

$$\text{gamma} := U_0$$

$$\text{beta} := \frac{\sqrt{U_1^{2} + U_2^{2}}}{U_0} \qquad \text{beta} = 0.866$$

$$L4\_1 := \begin{bmatrix} \text{gamma} & -\text{gamma}\cdot\text{beta}\cdot N_1 & -\text{gamma}\cdot\text{beta}\cdot N_2 \\ -\text{gamma}\cdot\text{beta}\cdot N_1 & 1 & 0 \\ -\text{gamma}\cdot\text{beta}\cdot N_2 & 0 & 1 \end{bmatrix}$$

$$L4\_2 := (\text{gamma} - 1)\cdot Pr \qquad L4 := L4\_1 + L4\_2$$

Here we have used the fact that the matrix of V->V_spatial is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$L4 = \begin{bmatrix} 2 & -1.414 & -1 \\ -1.414 & 1.667 & 0.471 \\ -1 & 0.471 & 1.333 \end{bmatrix} \qquad L3 = \begin{bmatrix} 2 & -1.414 & -1 \\ -1.414 & 1.667 & 0.471 \\ -1 & 0.471 & 1.333 \end{bmatrix}$$

As expected, they are equal.