Math. 460 (Fulling) 9 December 2013

Final Examination – Solutions

1. *(Hero worship – 40 pts.)* Which name is associated with each of these?

CHOICES

Kruskal
Einstein
Riemann
Kerr
Yang and Mills
Birkhoff
Hubble
Friedmann

(a) Rotating black holes

Kerr

(b) Uniqueness of spherical solutions

Birkhoff

(c) Dynamics of the expanding universe

Friedmann

(d) The curvature tensor

Riemann

(e) Expansion rate of the universe

Hubble

(f) The gravitational field equations

Einstein

(g) Non-Abelian gauge field theory

Yang and Mills

(h) Removal of the coordinate singularity in the Schwarzschild black-hole solution

Kruskal

2. *(60 pts.)* A two-dimensional model of the Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \,.$$

Notice that the expression manifestly makes sense for $-\infty < t < \infty$ and either $0 < r < 2M$ or $2M < r < \infty$ while having singularities at $r = 0$ and $r = 2M$.

(a) Find the equations of motion of a particle in this space-time (the geodesic equation).

The Lagrangian is

$$L = \frac{1}{2}\left[-\left(1-\frac{2M}{r}\right)\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2\right].$$

$$\frac{\partial L}{\partial \dot{t}} = -\left(1-\frac{2M}{r}\right)\dot{t}\,, \qquad \frac{\partial L}{\partial t} = 0\,,$$

so (if λ is the proper time)

$$\frac{d}{d\lambda}\left[\left(1-\frac{2M}{r}\right)\dot{t}\right] = 0\,,$$

which we should solve immediately to say that

$$\left(1-\frac{2M}{r}\right)\dot{t} = \text{constant}\,.$$

(This constant is the energy, $-p_0$, if the particle mass is 1 .)

$$\frac{\partial L}{\partial \dot{r}} = \left(1-\frac{2M}{r}\right)^{-1}\dot{r}\,, \qquad \frac{\partial L}{\partial r} = -\frac{M}{r^2}\dot{t}^2 - \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-2}\dot{r}^2\,.$$

Then

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{r}} = \left(1-\frac{2M}{r}\right)^{-1}\ddot{r} - \left(1-\frac{2M}{r}\right)^{-2}\left(\frac{2M}{r^2}\right)\dot{r}^2\,,$$

so the second equation of motion simplifies to

$$\ddot{r} = \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{M}{r^2}\right)(\dot{r}^2 - {p_0}^2)\,.$$

With some effort this can be shown equivalent to the simpler

$$\dot{r}^2 = {p_0}^2 - \left(1-\frac{2M}{r}\right)$$

obtained in Sec. 11.1 of Schutz by momentum analysis (with $L=0$).

(b) In the region $r > 2M$, introduce new coordinates by

$$u = -t + r + 2M\ln\left(\frac{r}{2M}-1\right)\,,$$
$$v = +t + r + 2M\ln\left(\frac{r}{2M}-1\right)\,.$$

(Warning: These are not the coordinates called u and v in Schutz's book and some of the original papers. The latter are those called T and R in the remark below.) Assume that the line element will have the form

$$ds^2 = f(r)\,du\,dv$$

and find f. (Note: This is easier than the more obvious question, "Find ds^2 in terms of u and v," but equivalent in the end.)

$$du = dt - dr - \frac{dr}{\frac{r}{2M} - 1} = dt - \frac{dr}{1 - \frac{2M}{r}}\,.$$

Similarly, $dv = dt + \dfrac{dr}{1 - \frac{2M}{r}}$. So

$$du\,dv = dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} = \left(1 - \frac{2M}{r}\right)^{-1} ds^2.$$

Thus $f(r) = \left(1 - \dfrac{2M}{r}\right)$.

Remark: You can take my word for it that you would find exactly the same result for $0 < r < 2M$ after keeping track of all the minus signs. If you then made the further transformations

$$U = e^{u/4M}\,\mathrm{sgn}\left(\frac{r}{2M} - 1\right), \qquad R = \tfrac{1}{2}(V + U),$$
$$V = e^{v/4M}, \qquad T = \tfrac{1}{2}(V - U),$$

you would arrive at

$$ds^2 = \frac{32M^3}{r}\, e^{-r/2M}(-dT^2 + dR^2)$$

with

$$R^2 - T^2 = e^{r/2M}\left(\frac{r}{2M} - 1\right).$$

All of this remains true in the full 4-dimensional space-time.

Checking part of the Remark:

$$dU\,dV = \pm(4M)^{-2} UV\,du\,dv\,.$$

where

$$UV = \pm e^{(u+v)/4M} = \pm e^{r/2M}\left|\frac{r}{2M} - 1\right| = e^{r/2M}\left(\frac{r}{2M} - 1\right).$$

So f now gets replaced by

$$16M^2(UV)^{-1} f(r) = \frac{32M^3}{r}\, e^{-r/2M}\left(1 - \frac{2M}{r}\right)^{1-1}\,.$$

(c) Taking the facts in the Remark as given, draw the famous *[insert name from Qu. 1(h)]* diagram in the $R{-}T$ plane and discuss it. What happens at $r = 2M$? What happens at $r = 0$? In each region, what variables are "time"? (Don't try to calculate curvature in the 2D model, but use any qualitative knowledge you have about the curvature in the 4D case.)

3. *(50 pts.)* The radiation-dominated, spatially flat cosmology is governed by three equations:

$$p = \tfrac{1}{3}\rho \qquad \text{(equation of state)}\,,$$

$$\frac{d}{dt}(\rho A^3) = -p\,\frac{d}{dt}A^3 \qquad \text{(conservation law)}\,,$$

$$\dot{A}^2 = \frac{8\pi}{3}\,\rho A^2 \qquad \text{(gravitational field equation)}\,.$$

(a) Why is this system called "radiation-dominated"? (In other words, where did that equation of state come from?)

It is appropriate to electromagnetic radiation, or, in quantum terms, massless particles such as photons (or tiny-mass particles such as neutrinos). For such particles the square of the energy equals the sum of the squares of the three momentum components. The pressure on a container wall is associated with the bouncing off of particles, and hence reversal of the perpendicular component of momentum. On average that amounts to one-third of the energy. Details are in most stat mech textbooks.

(b) Find ρ as a function of A. Hint: $\frac{d}{dt} = \frac{dA^3}{dt}\,\frac{d}{d(A^3)}$.

From the hint and the chain rule,

$$-\frac{1}{3}\,\rho = \frac{d}{dA^3}(\rho A^3) = A^3\,\frac{d\rho}{dA^3} + \rho\,.$$

So

$$\frac{d\rho}{\rho} = -\,\frac{4}{3}\,\frac{dA^3}{A^3}\,,$$

or

$$\ln\rho = -\,\frac{4}{3}\,\ln A^3 + c\,,$$

or

$$\rho = CA^{-4} \qquad (C > 0)\,.$$

(c) Find A as a function of t. Hint: The answer to (b) is some power of A.

$$\dot{A}^2 = \frac{8\pi C}{3}\,A^{-2}\,,$$

so

$$A\,dA = \pm\sqrt{\frac{8\pi C}{3}}\,dt\,.$$

I choose the positive sign to get an expanding solution and choose the next constant of integration to put the "big bang" at $t = 0$:

$$\frac{1}{2}\,A^2 = \sqrt{\frac{8\pi C}{3}}\,t\,.$$

More simply put, $A(t)$ is proportional to $\sqrt{t}$ with a constant whose value is an initial condition.

4. *(50 pts.)* Do **ONE** of these [(A) or (B)]. **If you try both, clearly indicate which (b) part you want graded. You can get credit for the extra (a) part.**

(A) Recall that under a non-Abelian gauge transformation $U(x)$ a connection form transforms by the law

$$\tilde{w}_\mu = U w_\mu U^{-1} - (\partial_\mu U) U^{-1} \ \textbf{ or } \ U(w_\mu - U^{-1}\partial_\mu U)U^{-1},$$

and also that the gauge field strength is defined by

$$Y_{\mu\nu} = w_{\nu,\mu} - w_{\mu,\nu} + [w_\mu, w_\nu].$$

(a) Explain why Y deserves to be called a "curvature tensor".

It represents the commutator of covariant derivatives in different directions. Alternatively, it describes the effect of parallel transport (of a field with nonvanishing non-Abelian charge) around a small closed path.

(b) Find the gauge transformation law for $\tilde{Y}_{\mu,\nu}$ (by either an abstract argument or a direct calculation). Hint: $\partial_\mu(U^{-1}) = -U^{-1}(\partial_\mu U)U^{-1}$.

For the abstract argument see the chapter from my book, p. 168.

Direct calculation (credit Steven Murray, in part):

$$\tilde{Y}_{\mu,\nu} = \tilde{w}_{\nu,\mu} - \tilde{w}_{\mu,\nu} + [\tilde{w}_\mu, \tilde{w}_\nu].$$

$$\begin{aligned}
\tilde{w}_{\nu,\mu} &= [U w_\nu U^{-1} - (\partial_\nu U)U^{-1}]_{,\mu} \\
&= \partial_\mu U w_\nu U^{-1} + U w_{\nu,\mu} U^{-1} + U w_\nu (\partial_\mu U^{-1}) - (\partial_\mu \partial_\nu U)U^{-1} - (\partial_\nu U)(\partial_\mu U^{-1}) \\
&= -(\partial_\mu \partial_\nu U)U^{-1} + U w_{\nu,\mu} U^{-1} + \partial_\mu U w_\nu U^{-1} - U w_\nu U^{-1}(\partial_\mu U)U^{-1} + (\partial_\nu U)U^{-1}(\partial_\mu U)U^{-1},
\end{aligned}$$

so

$$\begin{aligned}
\tilde{w}_{\nu,\mu} - \tilde{w}_{\mu,\nu} &= U(w_{\nu,\mu} - w_{\mu,\nu})U^{-1} \\
&\quad + \{\partial_\mu U w_\nu U^{-1} - U w_\nu U^{-1}(\partial_\mu U)U^{-1} + (\partial_\nu U)U^{-1}(\partial_\mu U)U^{-1} - [\mu \leftrightarrow \nu]\}.
\end{aligned}$$

Also,

$$\begin{aligned}
[\tilde{w}_\mu, \tilde{w}_\nu] &= \tilde{w}_\mu \tilde{w}_\nu - \tilde{w}_\nu \tilde{w}_\mu \\
&= U(w_\mu - U^{-1}\partial_\mu U)U^{-1} U(w_\nu - U^{-1}\partial_\nu U)U^{-1} - [\mu \leftrightarrow \nu] \\
&= U(w_\mu w_\nu - w_\nu w_\mu)U^{-1} \\
&\quad + \{-(\partial_\mu U)w_\nu U^{-1} - U w_\mu U^{-1}(\partial_\nu U)U^{-1} + (\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1} - [\mu \leftrightarrow \nu]\}.
\end{aligned}$$

Then all the messy terms cancel, leaving

$$\begin{aligned}
\tilde{Y}_{\mu,\nu} &= U(w_{\nu,\mu} - w_{\mu,\nu})U^{-1} + U[w_\mu, w_\nu]U^{-1} \\
&= U Y_{\mu,\nu} U^{-1}.
\end{aligned}$$

(B)

(a) In $G_{\mu\nu} = 8\pi T_{\mu\nu}$, what is G? Tell how to define it from the Ricci tensor.

(b) Prove that $G^{\mu\nu}{}_{;\nu} = 0$. (Start from the Bianchi identity relating first derivatives of the full curvature tensor.)

See book, pp. 164–165.