Math. 460 (Fulling) 11 October 2013

Test A – Solutions

1. *(48 pts.)* In the region $|t| < x$ of two-dimensional Minkowski space (flat space-time), introduce new coordinates (a, b) by

$$t = a \tan b\,, \qquad x = a \sec b = \frac{a}{\cos b}\,.$$

Help: $\frac{d}{db} \sec b = \sec b \tan b\,, \qquad \sec^2 b - \tan^2 b = 1\,, \qquad \frac{d}{du} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}}$

(a) Find the tangent vectors to the coordinate curves, $\vec{e}_a$ and $\vec{e}_b$.

Write $\vec{r} = \begin{pmatrix} t \\ x \end{pmatrix}$. Then

$$\vec{e}_a = \frac{d\vec{r}}{da} = \begin{pmatrix} \tan b \\ \sec b \end{pmatrix}, \qquad \vec{e}_b = \frac{d\vec{r}}{db} = \begin{pmatrix} a \sec^2 b \\ a \sec b \tan b \end{pmatrix}.$$

(b) Find the metric tensor, $\{g_{\mu\nu}\}$, (or, equivalently, the line element, ds^2) in the new coordinates.

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 \\ &= -\left(\frac{\partial t}{\partial a}\, da + \frac{\partial t}{\partial b}\, db\right)^2 + \left(\frac{\partial x}{\partial a}\, da + \frac{\partial x}{\partial b}\, db\right)^2 \\ &= \left(-\tan^2 b + \sec^2 b\right) da^2 + \left(-a^2 \sec^4 b + a^2 \sec^2 b \tan^2 b\right) db^2 \\ &\quad + \left(-a \tan b \sec^2 b + a \sec^2 b \tan b\right) da\, db \\ &= da^2 - a^2 \sec^2 b\, db^2\,. \end{aligned}$$

Remark: There is a function called the *Gudermannian*, $\operatorname{gd}(u) \equiv 2 \tan^{-1}(e^u) - \frac{\pi}{2}$, with the properties

$$\cosh u = \sec\big(\operatorname{gd}(u)\big)\,, \qquad \sinh u = \tan\big(\operatorname{gd}(u)\big)\,.$$

So these coordinates are the familiar hyperbolic coordinates with a nonlinear distortion of the time variable. To check: If $b = \operatorname{gd}(u)$, then

$$db = \frac{2e^u\, du}{1 + e^{2u}} = \frac{du}{\cosh u}\,,$$

so $\sec^2 b\, db^2 = du^2$ and hence $ds^2 = da^2 - a^2 du^2$.

(c) Find the Christoffel symbols ($\Gamma^a{}_{ab}$, etc.).

Method 1: Use $\dfrac{\partial \vec{e}_\mu}{\partial x^\nu} = \Gamma^\sigma_{\mu\nu}\vec{e}_\sigma$.

$$\frac{\partial \vec{e}_a}{\partial a} = 0\,, \qquad \frac{\partial \vec{e}_a}{\partial b} = \begin{pmatrix} \sec^2 b \\ \sec b \tan b \end{pmatrix} = \frac{1}{a}\,\vec{e}_b\,,$$

$$\frac{\partial \vec{e}_b}{\partial a} = \begin{pmatrix} \sec^2 b \\ \sec b \tan b \end{pmatrix} = \frac{1}{a}\,\vec{e}_b\,,$$

$$\frac{\partial \vec{e}_b}{\partial b} = a\begin{pmatrix} 2\sec^2 b \tan b \\ \sec b \tan^2 b + \sec^3 b \end{pmatrix} = a\sec b\begin{pmatrix} 2\tan b \sec b \\ \tan^2 b + \sec^2 b \end{pmatrix} = a\sec^2 b\,\vec{e}_a + \tan b\,\vec{e}_b\,.$$

Therefore,

$$\Gamma^a_{aa} = 0 = \Gamma^b_{aa}\,, \qquad \Gamma^a_{ab} = 0 = \Gamma^a_{ba}\,,$$

$$\Gamma^b_{ab} = \frac{1}{a} = \Gamma^b_{ba}\,, \qquad \Gamma^a_{bb} = a\sec^2 b\,, \qquad \Gamma^b_{bb} = \tan b\,.$$

Method 2: Get the same results more tediously from

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(g_{\mu\rho,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho}\right).$$

Method 3: Use $0 = \dfrac{d}{ds}\dfrac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \dfrac{\partial \mathcal{L}}{\partial x^\mu}$ with

$$\mathcal{L} = \frac{1}{2}\,g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{1}{2}\left(\dot{a}^2 - a^2\sec^2 b\,\dot{b}^2\right).$$

$$\frac{\partial \mathcal{L}}{\partial \dot{a}} = \dot{a}\,, \qquad \frac{\partial \mathcal{L}}{\partial a} = -a\sec^2 b\,\dot{b}^2\,,$$

so the first geodesic equation is

$$0 = \ddot{a} + a\sec^2 b\,\dot{b}^2 \;\Rightarrow\; \Gamma^a_{aa} = 0\,, \quad \Gamma^a_{ab} = 0\,, \quad \Gamma^a_{bb} = a\sec^2 b\,.$$

The other equation comes from

$$\frac{\partial \mathcal{L}}{\partial \dot{b}} = -a^2\sec^2 b\,\dot{b}\,, \qquad \frac{\partial \mathcal{L}}{\partial b} = -a^2\sec^2 b\tan^2 b\,\dot{b}^2\,,$$

hence (after division by $-\frac{1}{2}a^2\sec^2 b$)

$$0 = \ddot{b} + 2\,\frac{\dot{a}}{a}\,\dot{b} + \tan b\,\dot{b}^2 \;\Rightarrow\; \Gamma^b_{aa} = 0\,, \quad \Gamma^b_{ab} = \frac{1}{a}\,, \quad \Gamma^b_{bb} = \tan b\,.$$

(d) Find the normal one-forms to the coordinate "surfaces", $\tilde{d}a$ and $\tilde{d}b$ (also called $\tilde{E}^a$ and $\tilde{E}^b$).

Easy way: Invert the matrix whose columns you found in (a):

$$J \equiv \begin{pmatrix} \tan b & a \sec^2 b \\ \sec b & a \sec b \tan b \end{pmatrix} \Rightarrow J^{-1} = \frac{1}{\det J} \begin{pmatrix} a \sec b \tan b & -a \sec^2 b \\ -\sec b & \tan b \end{pmatrix},$$

and the determinant is $a \sec b\,(\tan^2 b - \sec^2 b) = -a \sec b$. Simplify and identify the normal one-forms as the rows of this matrix:

$$da = \frac{\partial a}{\partial t}\, dt + \frac{\partial a}{\partial x}\, dx = -\tan b\, dt + \sec b\, dx\,,$$

$$db = \frac{\partial b}{\partial t}\, dt + \frac{\partial b}{\partial x}\, dx = \frac{1}{a}\, dt - \frac{1}{a}\, \sin b\, dx\,.$$

Hard way: Solve for a and b .

$$a = \sqrt{x^2 - t^2}\,, \qquad b = \sin^{-1}\left(\frac{t}{x}\right).$$

$$\frac{\partial a}{\partial t} = \frac{-t}{\sqrt{x^2 - t^2}} = -\tan b\,, \qquad \frac{\partial a}{\partial x} = \frac{x}{\sqrt{x^2 - t^2}} = \sec b\,,$$

$$\frac{\partial b}{\partial t} = \frac{1/x}{\sqrt{1 - \left(\frac{t}{x}\right)^2}} = \frac{1}{\sqrt{x^2 - t^2}} = \frac{1}{a}\,, \qquad \frac{\partial b}{\partial x} = \frac{-t/x^2}{\sqrt{1 - \left(\frac{t}{x}\right)^2}} = \frac{-t/x}{\sqrt{x^2 - t^2}} = -\frac{\sin b}{a}\,.$$

Finish as before.

(e) Verify that the basis you found in (b) is dual to the basis you found in (a). (Explain what "dual" means in this context.)

We want $\tilde{d}x^\mu(\vec{e}_\nu) = \delta^\mu_\nu$ (in the new coordinate system). If you did (d) the easy way, then this is just a matter of checking that the answer really is the inverse. For example,

$$\begin{aligned} \tilde{d}a(\vec{e}_b) &= \big(\tan b\, dt + \sec b\, dx\big) \begin{pmatrix} a \sec^2 b \\ a \sec b \tan b \end{pmatrix} \\ &= a \sec b(-\tan b \sec b + \sec b \tan b) = 0 \end{aligned}$$

(the same as the first row of J^{-1} times the second column of J).

2. *(essay – 15 pts.)* In the Brans-Stewart periodic universe, an observer in motion ages more slowly than an observer at rest (as determined by comparison of clocks at their reunion). Yet each is always in uniform motion relative to the other, at the same speed. Explain why this does not contradict the "principle of relativity".

3. *(37 pts.)*

(a) Explain in modern language (multilinear functionals and all that) what a $\binom{1}{1}$ tensor is. (Call the tensor S.)

S is a bilinear function of one vector argument and one covector argument. That is, if $\mathcal{V}$ is the basic vector space of the problem, then for every $\vec{v} \in \mathcal{V}$ and every $\tilde{\omega} \in \mathcal{V}^*$, $S(\vec{v}, \tilde{\omega})$ is a real number, and for any real number r we have

$$S(r\vec{v}_1 + \vec{v}_2, \tilde{\omega}) = rS(\vec{v}_1, \tilde{\omega}) + S(\vec{v}_2, \tilde{\omega})$$

and the similar equation for linear combinations of covectors.

(b) If $\{L^{\overline{\mu}}{}_{\nu}\}$ is the matrix of a linear coordinate transformation of vectors ($\binom{1}{0}$ tensors), state the transformation law for the components, $\{S^{\mu}{}_{\nu}\}$, of a $\binom{1}{1}$ tensor.

$$S^{\overline{\mu}}{}_{\overline{\nu}} = L^{\overline{\mu}}{}_{\rho} L^{\sigma}{}_{\overline{\nu}} S^{\rho}{}_{\sigma} ,$$

where $L^{\sigma}{}_{\overline{\nu}}$ (because of the position of the barred index) is understood to be the inverse of the original L and the order of its indices implements an additional transpose operation. If we move that factor to the extreme right, we get the standard formula for similarity transformation of an operator, LSL^{-1}.

(c) Implement the transformation in (b) in this case:

- The vector space is the space of velocity vectors at the point $(a, b) = (2, \pi/4)$ in the space-time of Question 1.
- The linear transformation L maps coordinates with respect to the basis $\{\vec{e}_a, \vec{e}_b\}$ to coordinates with respect to the original Cartesian basis, $\{\vec{e}_t, \vec{e}_x\}$. (The Cartesian basis is the "barred" one.)
- The matrix representing S is $\begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}$.

(Just write down the correct numerical matrix product — don't do the arithmetic of the matrix multiplication.)

$L^{\overline{\mu}}{}_{\nu} = \dfrac{\partial x^{\overline{\mu}}}{\partial x^{\nu}}$, and from 1(a) we can read off

$$L = \begin{pmatrix} \frac{\partial t}{\partial a} & \frac{\partial t}{\partial b} \\ \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} \end{pmatrix} = \begin{pmatrix} \tan b & a \sec^2 b \\ \sec b & a \sec b \tan b \end{pmatrix}.$$

In other words, $L = J$ (as defined in the solution to 1(d) above). Now set $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{\pi}{4} \end{pmatrix}$, getting

$$L = \begin{pmatrix} 1 & 4 \\ \sqrt{2} & 2\sqrt{2} \end{pmatrix}, \qquad L^{-1} = \begin{pmatrix} -1 & \sqrt{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{4} \end{pmatrix}.$$

Alternatively, you can get the inverse from 1(d):

$$L^{-1} = \begin{pmatrix} \frac{\partial a}{\partial t} & \frac{\partial a}{\partial x} \\ \frac{\partial b}{\partial t} & \frac{\partial b}{\partial x} \end{pmatrix} = \begin{pmatrix} -\tan b & \sec b \\ \frac{1}{a} & -\frac{\sin b}{a} \end{pmatrix} = \begin{pmatrix} -1 & \sqrt{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{4} \end{pmatrix}.$$

Now we can use these matrices in the tensor transformation formula in (b), or, probably simpler, use the similarity transformation

$$S \text{ (barred)} = LSL^{-1} ,$$

where the three matrices are given above and I will not recopy them since that forces a 5th page.