Some details about the twin paradox

First review the standard situation of two frames in relative motion at speed v. The t' axis (path of the moving observer) has slope $1/v$. The x' axis and all other equal-time hypersurfaces of the moving observer have slope v.

Second, consider the standard twin scenario as graphed by Lowry (and Schutz in the appendix to Chapter 1).

Let the starting point be $t = t' = 0$, $x = x' = 0$, and let the point of return be $(0, t)$ in the stationary frame. The stationary observer attributes a time dilation to the moving clock:

$$t' = \frac{t}{\gamma}$$

where $\gamma = (1 - v^2)^{-1/2} > 1$. The moving observer attributes to the stationary clock a similar dilation plus a gap:

$$t = \frac{t'}{\gamma} + \epsilon.$$

Let's calculate ϵ: Consistency requires

$$\epsilon = t - \frac{t'}{\gamma} = t\left(1 - \frac{1}{\gamma^2}\right) = v^2 t,$$

since

$$\frac{1}{\gamma^2} = 1 - v^2.$$

To see this a different way, let $T = t/2$ and observe that the distance traveled outward is $L = vT$. Therefore, since $t' = \text{const}$ surfaces have slope v. the half-gap is $\tau = vL = v^2 T$. Thus $\epsilon = 2\tau = 2v^2 T = v^2 t$, as claimed.

Third, consider the Brans–Stewart model with circumference 1.

Note that the labeling of the $t' = \text{const}$ surfaces is ambiguous: The line with $t' = \epsilon/\gamma$ is also a continuation of the line $t' = 0$. (At this point in the diagram, ϵ is an arbitrary time, but we shall see that it can be identified with the ϵ in the previous discussion.)

Follow the moving observer around the cylinder back to the starting point (the x axis). In continuously varying coordinates this happens at $x = 1$, not $x = 0$. The distance traveled is vt, but it also equals 1, so we have

$$t = \frac{1}{v}\,.$$

Again we can say that from the stationary point of view, the elapsed times satisfy $t' = t/\gamma$, and from the moving point of view they must satisfy $t = t'/\gamma + \epsilon$ for some gap ϵ, although the geometrical origin of the gap may not be obvious yet. So by the same algebra as in the Lowry case, $\epsilon = v^2 t$. But in the present case that implies

$$\epsilon = v.$$

How can we understand this result? Follow the x' axis ($t' = 0$ curve) Around the cylinder. In stationary coordinates the distance "traveled" by this superluminal path is 1, but its "speed" is $1/v$. Therefore, $1 = t/v$, or

$$t = v = \epsilon.$$

Thus ϵ is the spacing (in t, not t') of the helical winding of the x' axis. This vindicates the labeling of ϵ in the diagram, and it shows that the gap term in the moving observer's calculation of the total time of his trip in the stationary observer's clock comes from jumping from one labeling of some $t' = \text{const}$ curve to the next (from t' to $t' + \epsilon/\gamma$).

Another way of looking at it is to use (5) of the Brans–Stewart paper, specialized to $n = -1$. This is the claim that the coordinates

$$(x', t') \quad \text{and} \quad (x' - \gamma, t' + \gamma v)$$

represent the same event (space-time point). We can check this from the Lorentz transformation (inverted from (4) of Brans–Stewart)

$$\begin{aligned} x =& \gamma(x' + vt'), \\ t =& \gamma(t' + vx'). \end{aligned}$$

We get

$$\begin{aligned} x_{\text{new}} = \gamma(x' - \gamma + v(t' + \gamma v)) = \gamma^2(v^2 - 1) + \gamma(x' + vt') = x - 1, \\ t_{\text{new}} = \gamma(t' + \gamma v + v(x' - \gamma)) = \gamma(t' + vx') = t. \end{aligned}$$

Now, if

$$t' = \frac{t}{\gamma}\,, \tag{$*$}$$

then

$$t' + \gamma\epsilon = \gamma\left(\frac{t'}{\gamma} + \epsilon\right),$$

or

$$t' + \gamma\epsilon = \gamma t. \tag{\#}$$

Comparing ($*$) and (#), we see that the γ has "flipped" exactly as needed to make the time dilation formula consistent for each observer. This is essentially the same as our first calculation of ϵ, but stated on the time scale of t', not t.