Math. 467 (Fulling) 10 December 2010

Final Examination

Name: ______________________________

1. *(20 pts.)* Rearrange these names into historical order, earliest to latest:

Thales, Klein, Euclid, Lambert, Lobachevsky

Thales, Euclid, Lambert, Lobachevsky, Klein

2. *(30 pts.)* State the **converse** of the Alternate Internal Angle Theorem, and prove that it is equivalent to the Euclidean parallel postulate (**HE**).

[See Greeberg pp. 164 and 175 and the student-written proof of Prop. 4.8 on the web site.]

3. *(Multiple choice – each 5 pts.)*

   (a) Poincaré's disk model is better than Klein's in which respect?
   (A) It satisfies more of the axioms.
   (B) It is *conformal*, meaning that it represents angles accurately.
   (C) Its lines are straight Euclidean lines.
   (D) [none of these]

B

   (b) Consider these propositions:
   1. Two (distinct) points determine a unique line.
   2. Two (distinct) nonparallel lines determine a unique point.

   Which of the following statements is **FALSE**?
   (A) We assumed one of these as an axiom and proved the other from it.
   (B) The uniqueness claim in 1. can easily be proved from 2.
   (C) In elliptic geometry, this pair of facts demonstrates "duality".
   (D) 2. is one of Hilbert's axioms.
   (E) Euclid's first postulate (**EI**) is essentially 1.

D

   (c) In hyperbolic geometry the ratio of the circumference of a circle to its radius is
   (A) greater than 2π.
   (B) equal to 2π.
   (C) less than 2π.

A

   (d) Hyperbolic lengths are different from Euclidean lengths in the
   (A) Klein model.
   (B) Poincaré disk model.
   (C) Poincaré half-plane model.
   (D) [all of these]
   (E) [none of these]

D

(e) Which of these statements is **FALSE** in hyperbolic geometry?
- (A) Rectangles do not exist.
- (B) Two parallel lines always have a common perpendicular.
- (C) Summit angles of Saccheri quadrilaterals are acute.
- (D) Every valid theorem of neutral geometry remains valid.
- (E) Every ray into the interior of an angle intersects any segment joining the two sides of the angle.

B

(f) We needed to prove the alternate interior angle theorem and exterior angle theorem before proving the congruence criterion
- (A) SSS
- (B) ASA
- (C) SAA
- (D) SSA
- (D) SAS

C

(g) Which statement is equivalent to $p \vee \neg q$?
- (A) $\neg p \wedge q$
- (B) $p \Rightarrow q$
- (C) $q \Rightarrow p$
- (D) $q \vee \neg p$
- (E) $\neg(\neg p \vee q)$

C

(h) Which of these statements is **TRUE**? If someone produces a proof of the Euclidean parallel postulate within neutral geometry,
- (A) the author will be instantly acclaimed as a genius.
- (B) it will show that Euclidean geometry is consistent.
- (C) it will show that Euclidean geometry is inconsistent.
- (D) it will show that the proof by Gergonne was logically valid after all.
- (E) it will show that one of the proofs by Legendre was logically valid after all.

C [With respect to (A), the author will be instantly rejected as a crank, even if he turns out in the end to be correct.]

(i) In this course, a *right angle* is **defined** as one that
- (A) is congruent to its supplement.
- (B) contains 180°.
- (C) is a base angle of a Saccheri quadrilateral.
- (D) is the foot of a perpendicular line.

A

(j) Similar triangles are congruent
  (A) always.
  (B) never: this is the notoriously fallacious "AAA" congruence criterion.
  (C) in hyperbolic geometry.
  (D) only in Dehn models built on non-Archimedean fields.

C

4. *(25 pts.)* Recall *Wallis's Axiom*:

   Given any triangle ΔABC and any segment DE, there exists a triangle ΔDEF having DE as one of its sides such that $\Delta\mathrm{ABC} \sim \Delta\mathrm{DEF}$ (that is, $\angle\mathrm{A} \cong \angle\mathrm{D}$, $\angle\mathrm{B} \cong \angle\mathrm{E}$, $\angle\mathrm{C} \cong \angle\mathrm{F}$

   Prove that Euclid's 5th postulate (**EV**) implies Wallis's axiom. *Hint:* Use the fact that the angles of a triangle sum to 180° if **EV** holds.

[This is Exercise 5-3(a) in Greenberg, p. 229. The proof is clear from the figure and hint given there.]

5. *(25 pts.)* Do **ONE** of these: (Use blank sheet.) **15 pts. extra credit for doing both**

   (A) In the Poincaré disk model, the formula for arc length is

$$ds^2 = \frac{4(dx^2 + dy^2)}{[1 - (x^2 + y^2)]^2}\,.$$

   Introduce polar coordinates (ρ, θ) in the (x, y) plane and show that the further coordinate transformation

$$\rho = \tanh\left(\frac{r}{2}\right)$$

   converts the arc length to

$$ds^2 = dr^2 + \sinh^2 r\, d\theta^2\,.$$

   Explain why this result justifies the vague claim that hyperbolic geometry describes "a sphere of imaginary radius".

   Hint:
$$\tanh^{-1} y = \frac{1}{2}\ln\frac{1+y}{1-y}\,.$$

   (B) Suppose that A and B are points in the Poincaré disk and that P and Q (points on the circle bounding the disk) are the ends of the Poincaré line through A and B. The *cross-ratio* is defined as

$$(\mathrm{AB}, \mathrm{PQ}) \equiv \frac{\overline{\mathrm{AP}}}{\overline{\mathrm{AQ}}}\,\frac{\overline{\mathrm{BQ}}}{\overline{\mathrm{BP}}}\,,$$

   where, for instance, $\overline{\mathrm{AB}}$ is the *Euclidean* distance between A and B. Show how to use the cross-ratio to define a distance (length) function in the hyperbolic geometry that is *additive* in the sense that

$$d(\mathrm{AB}) = d(\mathrm{AC}) + d(\mathrm{CB})$$

   when $\mathrm{A} * \mathrm{C} * \mathrm{B}$ along a Poincaré line. (Verify that it is indeed additive.)