Math. 467W (Fulling) 13 December 2010

Final Examination – Solutions

Name: ______________________________

1. *(20 pts.)* Rearrange these names into historical order, earliest to latest:

Proclus, Hilbert, Euclid, Saccheri, Bolyai (the son)

Euclid, Proclus, Saccheri, Bolyai, Hilbert

2. *(Essay – 30 pts.)* State the Alternate Internal Angle Theorem, and explain (by citing axioms) why it fails in spherical and elliptic geometry.

[See Greenberg, pp. 162–163, and related pages of my lecture notes. The relevant axioms are I-1 (false in spherical), B-3 (false in both), B-4 (false in elliptic).]

3. *(Multiple choice – each 5 pts.)*

(a) Which of these statements is **TRUE** in hyperbolic geometry?
  (A) Rectangles exist, but not every Lambert quadrilateral is a rectangle.
  (B) Two parallel lines always have a common perpendicular.
  (C) Summit angles of Saccheri quadrilaterals are acute.
  (D) Not every valid theorem of neutral geometry remains valid.
  (E) Every point in the interior of an angle lies on a segment joining the two sides of the angle.

C

(b) Consider these propositions:
  1. Two (distinct) points determine a unique line.
  2. Two (distinct) nonparallel lines determine a unique point.
  Which of the following statements is **FALSE**?
  (A) We assumed one of these as an axiom and proved the other from it.
  (B) The uniqueness claim in 1. can easily be proved from 2.
  (C) In elliptic geometry, this pair of facts demonstrates “duality”.
  (D) 2. is one of Hilbert’s axioms.
  (E) Euclid’s first postulate (**EI**) is essentially 1.

D

(c) Which statement is equivalent to $p \vee \neg q$?
  (A) $\neg p \wedge q$
  (B) $q \Rightarrow p$
  (C) $p \Rightarrow q$
  (D) $\neg(\neg p \vee q)$
  (E) $q \vee \neg p$

B

(d) Hyperbolic angles are different from Euclidean angles in the
- (A) Klein model.
- (B) Poincaré disk model.
- (C) Poincaré half-plane model.
- (D) [all of these]
- (E) [none of these]

A

(e) In hyperbolic geometry the ratio of the circumference of a circle to its radius is
- (A) greater than 2π.
- (B) equal to 2π.
- (C) less than 2π.

A

4. *(Essay – 25 pts.)* Explain (in some detail, providing background information) the statement:

If you succeed in proving the parallel postulate, you will have shown that Euclidean geometry is *inconsistent*.

[See Greenberg pp. 289–293 and 72–79 and related pages of my lecture notes.]

5. *(25 pts.)* Recall *Wallis's Axiom*:

Given any triangle ΔABC and any segment DE, there exists a triangle ΔDEF having DE as one of its sides such that ΔABC $\sim$ ΔDEF (that is, $\angle$A $\cong$ $\angle$D, $\angle$B $\cong$ $\angle$E, $\angle$C $\cong$ $\angle$F).

Prove that Wallis's axiom implies the Euclidean parallel postulate (**HE**).

[See Greenberg pp. 216–217.]

6. *(25 pts.)* Do **ONE** of these: (Use blank sheet.) **15 pts. extra credit for doing both**

(A) In the Poincaré disk model, the formula for arc length is

$$ds^2 = \frac{4(dx^2 + dy^2)}{[1 - (x^2 + y^2)]^2} \,.$$

Introduce polar coordinates (ρ, θ) in the (x, y) plane and show that the further coordinate transformation

$$\rho = \tanh\left(\frac{r}{2}\right)$$

converts the arc length to

$$ds^2 = dr^2 + \sinh^2 r \, d\theta^2 \,.$$

Explain why this result justifies the vague claim that hyperbolic geometry describes "a sphere of imaginary radius".

Hint:

$$\tanh^{-1} y = \frac{1}{2} \ln \frac{1+y}{1-y} \,.$$

(B) Suppose that A and B are points in the Poincaré disk and that P and Q (points on the circle bounding the disk) are the ends of the Poincaré line through A and B. The *cross-ratio* is defined as

$$(\mathrm{AB}, \mathrm{PQ}) \equiv \frac{\overline{\mathrm{AP}}}{\overline{\mathrm{AQ}}} \frac{\overline{\mathrm{BQ}}}{\overline{\mathrm{BP}}},$$

where, for instance, $\overline{\mathrm{AB}}$ is the *Euclidean* distance between A and B. Show how to use the cross-ratio to define a distance (length) function in the hyperbolic geometry that is *additive* in the sense that

$$d(\mathrm{AB}) = d(\mathrm{AC}) + d(\mathrm{CB})$$

when $\mathrm{A} * \mathrm{C} * \mathrm{B}$ along a Poincaré line. (Verify that it is indeed additive.)