Math. 467 (Fulling) 9 December 2013

Final Examination – Solutions

Name: ____

1. *(Multiple choice – each 5 pts.)*

   (a) The notorious “warning” on p. 115 (about points interior to an angle)
   - (A) turned out to be unnecessary, once the “hypothesis of the obtuse angle” was excluded.
   - (B) indicated a contradiction of the crossbar theorem.
   - (C) became obviously justified when straight lines in the Klein model were examined.
   - (D) remains an unsettled question in geometry.

C

   (b) Which of these remarks about “models” is **NOT** true?
   - (A) Models of the Hilbert I axioms exist with finitely many points.
   - (B) Models of the Hilbert B axioms require infinitely many points.
   - (C) Models are crucial in proving the independence of the parallel postulate.
   - (D) In a good model the basic terms of a theory may not be reinterpreted in unexpected ways.

D

   (c) The property of Beltrami’s pseudosphere model that makes it unique is
   - (A) Its lines are actually straight lines in the Euclidean sense.
   - (B) It can be realized by a 3-dimensional object in physical Euclidean space.
   - (C) It represents angles accurately (conformally).
   - (D) It represents the entire hyperbolic plane, not just a portion of it.

B

   (d) The proof that “every triangle is isosceles”
   - (A) is valid in hyperbolic geometry but not in Euclidean.
   - (B) is a fallacy that confused the famous mathematicians Legendre and Gergonne.
   - (C) is valid in elliptic geometry but not in the Hilbert axioms.
   - (D) hinges on a misleading diagram.

D

   (e) In “neutral geometry” it is possible to prove
   - (A) Parallel lines exist.
   - (B) The sum of the angles in a triangle is 180°.
   - (C) If a line intersects one of two parallel lines, then it intersects the other.
   - (D) Rectangles exist.

A

2. *(30 pts.)* Rearrange these names into historical order, earliest to latest:

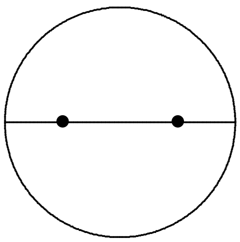
   Lobachevsky, Euclid, Pythagoras, Proclus, Hilbert, Saccheri

Pythagoras, Euclid, Proclus, Saccheri, Lobachevsky, Hilbert

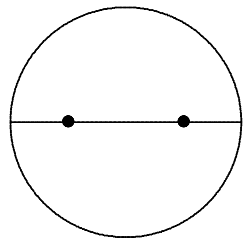
3. *(Essay – 35 pts.) [moved to end because of page break problem in Qu. 4]*

4. *(40 pts.)* For the Klein disk model of hyperbolic geometry (the one where lines are straight),

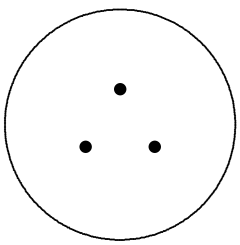
   (a) Verify the 3 Hilbert incidence axioms, the 4 betweenness axioms, and the hyperbolic property (in the form "$\exists(l, \mathrm{P})$ [more than one line parallel to l passes through P]", also known as $\neg$HE). Do this by drawing pictures in the disks provided and adding enough words of explanation to make the argument convincing.



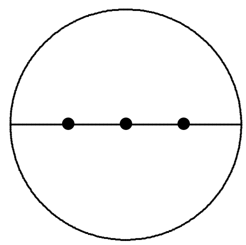
I-1
(points given)



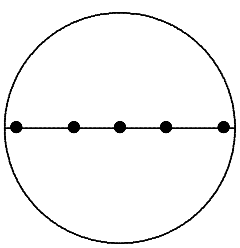
I-2
(line given)
Use Euclidean
B-2.



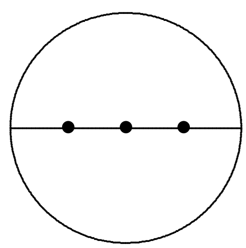
I-3



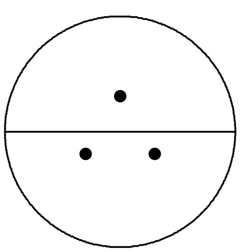
B-1
(Direction does
not matter . . .)



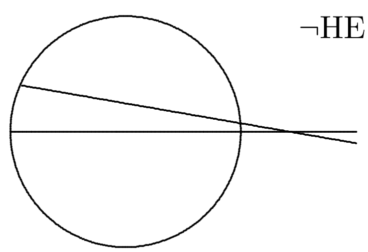
B-2
(two points
given)



B-3
(. . . but order
does!)



B-4
(2 sides of
given line)



$\neg$HE

The point is that we can use points and lines from the Euclidean plane $\mathbf{R}^2$, shortening the lines to chords. The only nontrivial part is assuring that the relevant points can be taken to stay inside the disk. (That's precisely what does *not* happen for HE!) For instance, repeated use of the Euclidean B-2 shows that the outermost points in the Klein B-2 can be chosen arbitrarily close to the given points, hence inside the circle.

(b) Discuss briefly how we know that the congruence axioms are also true in this (Klein's) model.

The congruence axioms can be proved (with difficulty) in Poincaré's disk model. There is a one-to-one mapping from Poincaré's disk to Klein's, and the segment and angle congruence relations are *defined* in the Klein model to be those inherited from the Poincaré model, so they satisfy the same axioms.

5. *(20 pts.)* State and prove **ONE** of these. **If you try both, clearly indicate which one you want graded.**

(A) the exterior angle theorem

See pp. 164–165.

(B) Under a suitable hypothesis (which you need to state), the sum of the angles in a triangle is 180°.

See p. 175. The hypothesis is HE.

3. *(Essay – 35 pts.)*

State the Alternate Interior Angle Theorem **and** its converse. Tell which of them can be proved from the Hilbert IBC axioms and which cannot. Explain which of them excludes elliptic geometry and which excludes hyperbolic geometry. Feel free to add comments on the significance of these theorems.

See pp. 162–164, 166, 175.